

# Games, graphs, and machines



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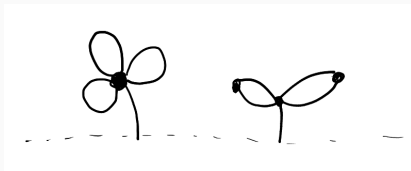
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Find the Grundy value of *Hackenbush* starting with the following.



Rules:

- Erase one edge on each turn.
- Any part of the graph not connected to the ground is auto-erased.
- The player who cannot make a move loses.

## Stable equivalence

Say  $G_1 \sim G_2$  if for all games  $H$ , both  $G_1 + H$  and  $G_2 + H$  have the same outcome (both  $N$  or both  $P$ ).

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# The Sprague-Grundy theorem

**Theorem.** Two games are equivalent if and only if they have the same Grundy value.

# Why?

One direction:  $\text{Grundy}(G_1) = \text{Grundy}(G_2) \implies G_1 \sim G_2$ .

# Why?

Other direction:  $G_1 \sim G_2 \implies \text{Grundy}(G_1) = \text{Grundy}(G_2)$ .

# Everything is (equivalent) to Nim

Every (impartial) game is equivalent to a (one-pile) Nim game.