

Games, graphs, and machines

Equivalence relations

July 28, 2025

Equivalence relation (reminder)

R is an equivalence relation on a set S if:

Reflexive For all $a \in S$, we have $(a, a) \in R$.

Symmetric If $(a, b) \in R$, then $(b, a) \in R$.

Transitive If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Equivalence relation or not?

Is \sim an equivalence relation on S ?

1. $S = \mathbb{R}$ and $a \sim b$ if $|a - b| < 1$.

2. $S =$ States of a chess-board and $a \sim b$ if we can reach b from a by a sequence of legal moves.

Equivalence relation or not?

Is \sim an equivalence relation on S ?

1. $S = \text{Pow}(A)$ and $A \sim B$ if $|A| = |B|$.
2. $S = \text{Pow}(A)$ and $A \sim B$ if $A \subset B$.
3. $S = \text{Pow}(A)$ and $A \sim B$ if $A \cap B \neq \emptyset$.

Graph of a relation

Let S be a set. It is helpful to visualise a relation by drawing a vertex for each element of S and an edge $a \rightarrow b$ if (a, b) are related. The resulting diagram is called a *graph*.

Take $S = \text{Pow}(1, 2)$ and draw the graph of \subset .

Graphical interpretation

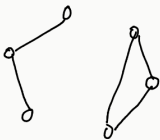
How are the following properties reflected in the graph?

- Reflexivity

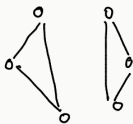
- Symmetry

Graph of an equivalence relation

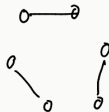
Which of the following is the graph of an equivalence relation?
(Self-loops and arrows omitted)



1



2



3

Equivalence classes

Let \sim be the relation on \mathbb{R}^2 defined by

$$(a, b) \sim (c, d) \text{ if } ab = cd.$$

Describe the equivalence classes.

How do you explain this?

Let $E(n, k)$ be the number of equivalence relations on an n -element set with k equivalence classes.

Let $F(n, k)$ be the number of surjections from an n -element set to a k element set.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	3	7	15	31	63	127	255	511	1023	2	6	14	30	62	126	254	510	1022		
1	6	25	90	301	966	3025	9330	28501		6	36	150	540	1806	5796	18150	55980			
1	10	65	350	1701	7770	34105	145750			24	240	1560	8400	40824	186480	818520				
1	15	140	1050	6951	42525	246730				120	1800	16800	126000	834120	5103000					
1	21	266	2646	22827	179487					720	15120	191520	1905120	16435440						
1	28	462	5880	63987						5040	141120	2328480	29635200							
1	36	750	11880							40320	1451520	30240000								
1	45	1155								362880	16329600									
1	55									3628800										
1																				

What do you observe?