**ASSIGNMENT 5 (DUE 29 AUGUST 2025)**

**MATH2301, SEMESTER 2, 2025**

## 1. General remarks

(1) The assignment is due on gradescope.
(2) Please read the academic integrity policy for assignments. Remember that if you want an extension, you must ask at least 24 hours ahead of the deadline.
(3) The words "show" and "prove" are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.
(4) If you are having trouble with any of the points, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

## 2. Problems

2.1. **Problem.** A directed graph is called *strongly connected* if for every $i$ and $j$, there is a path from vertex $i$ to vertex $j$ (we allow length 0 paths).

Select true or false.

(1) A graph with adjacency matrix $A$ is strongly connected if and only if for some $n$, the boolean power $A^{*n}$ has all entries equal to 1.
(2) A graph with adjacency matrix $A$ is strongly connected if and only if for some $n$, the boolean sum

$$I + A + A^{*2} + \cdots + A^{*n}$$

has all entries equal to 1.

If the statement is true, you should be able to justify it. Otherwise, you should be able to give a counter-example. But you need not turn in the justification or counter-examples.

2.2. **Problem.** Let $G$ be the directed graph of a relation, and let $A$ be the boolean adjacency matrix of $G$. Consider the following statements.

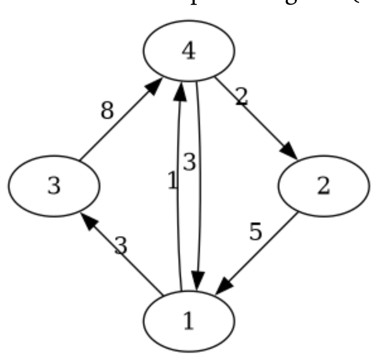
(1) If $A = A + A^{*2}$, then the relation is transitive.
(2) If the relation is transitive, then $A = A + A^{*2}$.

Decide whether (1) or (2) or both are true. Justify your answers.

Remember that if you think a statement is false, it is enough to provide one example where it fails. If you think it is true, you must give a reasoning that works for all examples.

*Caution:* There is no subtraction in boolean arithmetic. So if $x + y = x + z$, then you cannot conclude that $y = z$.

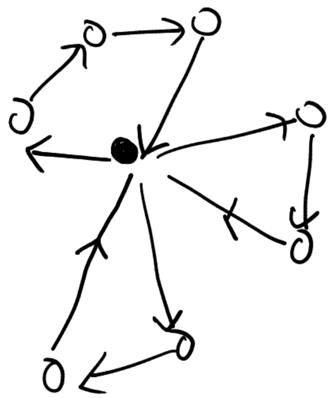
2.3. **Problem.** Find (using weighted adjacency matrices) the minimum cost of paths between any pair of vertices in the following graph. Assume that each vertex has loops of length 0 (not shown).



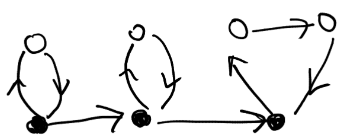
2.4. **Problem.** Let $G$ be a graph and $A$ its boolean adjacency matrix. Let $p$ be the period of the eventually periodic sequence $A, A^{*2}, A^{*3}, \ldots$.

Through experiments, try to understand $p$ for the following kinds of graphs:

**Windmills/flowers:** one central vertex (black) surrounded by multiple cycles of varying lengths. For example, below is a windmill with 3 cycles of length 4, 3, 3.



**Trains:** A backbone (black) carrying cycles of varying lengths. For example, below is a train with cycles of length 2, 2, 3.



(The black/white colouring is only for illustration, and has no other importance.)