

WEEK 4 WORKSHOP
MATH2301, SEMESTER 2, 2025

1. MATRIX MULTIPLICATION

1.1. **Warm-up.** Take two 2×2 matrices A and B of your choice and calculate AB . Try to find a pairs such that

(1) $AB \neq BA$

(2) $AB = 0$ but $A \neq 0$ and $B \neq 0$ (here 0 means the 0 matrix, which is the matrix all of whose entries are 0.)

Solution. Virtually any two matrices will satisfy $AB \neq BA$. For $AB = 0$ but $A \neq 0$ and $B \neq 0$, take $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

1.2. **The identity matrix.** The $n \times n$ identity matrix has 1's on the diagonal and 0's everywhere else. It is denoted by I_n or $I_{n \times n}$ or just I if the n is clear from the context. Convince yourself that for any $n \times n$ matrix A , we have

$$A \cdot I_n = I_n \cdot A = A.$$

Solution. Left to you.

2. ADJACENCY MATRICES

2.1. **Poset graphs.** Given a poset S , we can construct a directed graph $G(S)$ as follows. The vertices of $G(S)$ are the elements of S , and we draw an edge $a \rightarrow b$ if $a \leq b$.

Draw the graph $G(S)$ and write the adjacency matrix for the following S

(1) $S = \{1, 2, 3\}$ with the usual order.

(2) S is the divisor poset of 12.

Solution.

(1) The edges here are self-loops and $1 \rightarrow 2, 1 \rightarrow 3, 2 \rightarrow 3$. So the adjacency matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(2) The edges here are self-loops $1 \rightarrow$ everything and $2 \rightarrow 4, 2 \rightarrow 6, 2 \rightarrow 12, 3 \rightarrow 6, 3 \rightarrow 12, 4 \rightarrow 12, 6 \rightarrow 12$. So, if we number the divisors in order (1, 2, 3, 4, 6, 12) then the adjacency matrix is

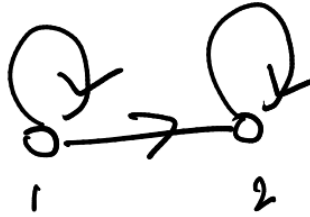
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. PATH COUNTING

3.1. **Warm-up.** Interpret $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ as the adjacency matrix of a graph. Calculate A^n for $n = 1, 2, 3, 4, \dots$. Can you explain what you see in terms of path counting?

Solution. We get $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

The entries of A^n are counting the paths of length n in the following graph.



There is exactly one path of length n from 1 to 1 and 2 to 2 (self-loop n times). There are exactly n paths of length n from 1 to 2: take k self-loops at 1, take the edge to 2, and $n - 1 - k$ self-loops at 2, for any value of $k = 0, \dots, n - 1$.

3.2. **Poset matrices.** Let S be the poset and A the adjacency matrix from the problem above on poset graphs.

(1) For $S = \{1, 2, 3\}$, find $(A - I)^3$. Can you explain the result?

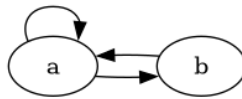
(2) For S equal to the divisor poset of 12, what do you think is the smallest positive n such that $(A - I)^n = 0$?

Solution. $A - I$ is the adjacency matrix of the poset graph with self-loops removed.

(1) There are no paths of length 3 in this graph, so $(A - I)^3 = 0$.

(2) We have to find the smallest n such that there are no paths of length n . That n is 4, I think. (There are length 3 paths like $1 \rightarrow 2 \rightarrow 4 \rightarrow 12$ or $1 \rightarrow 3 \rightarrow 6 \rightarrow 12$, but not length 4 paths).

3.3. **Rabbits?** Consider the following graph.



Calculate for a few values of k the number of length k paths from a to itself. Can you find (and perhaps prove!) a pattern?

Solution. If we order the vertices as a, b , then the adjacency matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and taking powers gives the Fibonacci sequence in the top-left spot. (Recall that the Fibonacci sequence starts with 1, 1, such that the next number is always the sum of the previous two). I won't write out the proof here, but happy to discuss if you are interested!

"Rabbits" is a reference to Fibonacci's puzzle from 1202 that led to the famous sequence (look it up).