Games, graphs, and machines

Partial orders 1

August 4, 2025

Partial orders

A *partial order* is a relation that is:

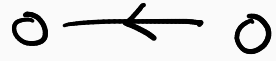
1. Reflexive
2. Anti-symmetric
3. Transitive

motivated by $\leq$

Anti symmetric : $(a,b) \in R \quad (b,a) \in R \Rightarrow a = b$.

If $a \neq b$ then at least one (a,b) or $(b,a) \notin R$.

OR



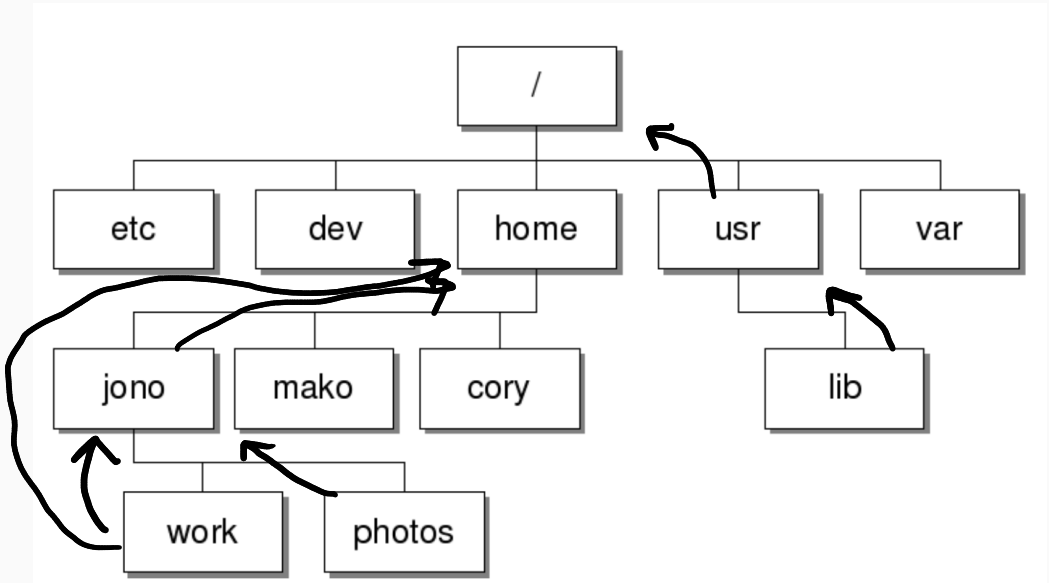
OR

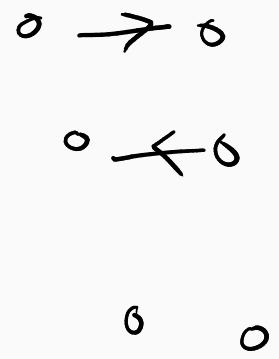
NOT :

Partial orders from real life

S = Directories

$\lesssim$





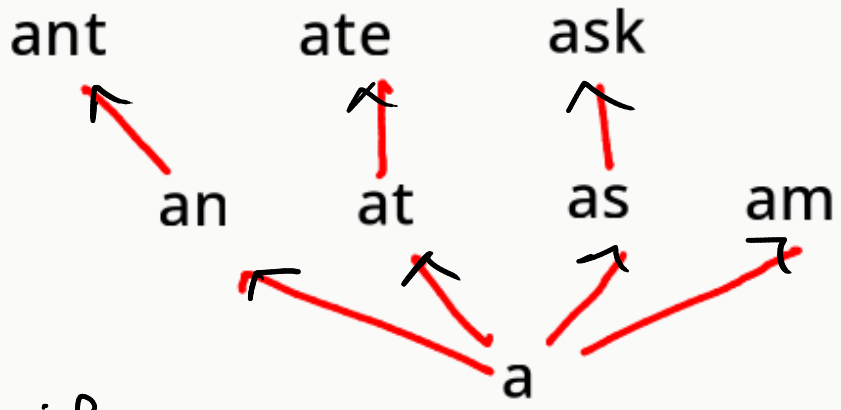
$a \lesssim b$ if a is contained in b

e.g. photos $\lesssim$ /

"Partial": unrelated pairs allowed!

total order: NO unrelated pairs

Partial orders from real life



"Hasse diagram"

$w_1 \preceq w_2$ if w_1 is a prefix of w_2

w_2 begins with w_1

Simplifications: ① Self loops omitted

② Edges implied by transitivity omitted.

Partial orders from real life

Non int. diagonals.

$a \preceq b$ if $a \supseteq b$

Paper, Scissors, Rock: Partial order?

- Paper $\preceq$ Scissors
- Scissors $\preceq$ Rock
- Rock $\preceq$ Paper

Not a partial order

Convention. $\sim$ or $\equiv$ for eqv. rel (similar to $=$)

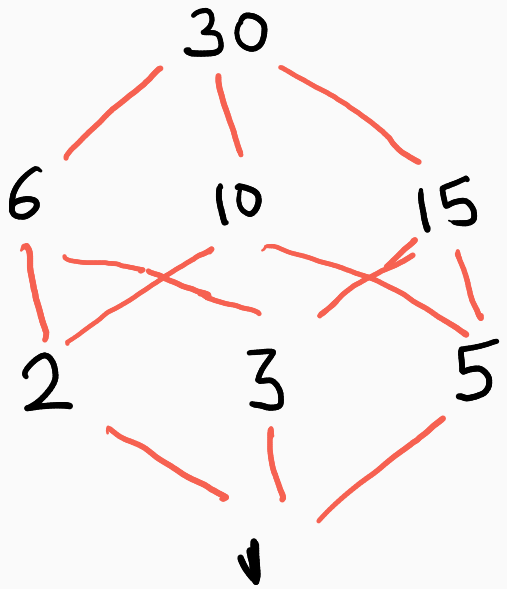
$\preceq$ or $\succeq$ for partial orders (similar to $\leq$)

Divisor poset

Let S be the set of divisors of 30; so $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Say $a \preceq b$ if a divides b.

What is the Hasse diagram?



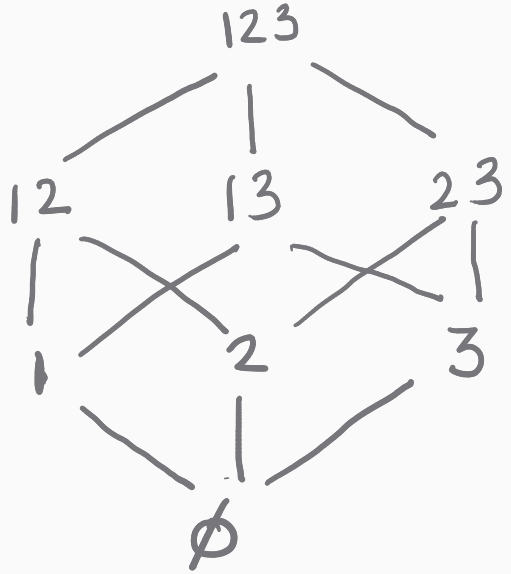
Subset poset

Let $S = \mathrm{Pow}(\{1, 2, 3\})$.

Say $A \preceq B$ if $A \subseteq B$.

What is the Hasse diagram?

$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$
$\{2,3\}$ $\{1,3\}$, $\{1,2,3\}$



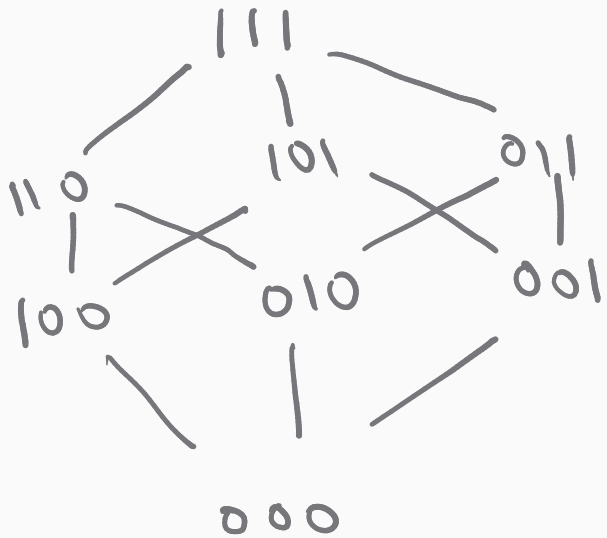
String poset

Let S be the set of binary (0 and 1) strings of length three.

Say $a_1a_2a_3 \preceq b_1b_2b_3$ if $a_1 \leq b_1$ and $a_2 \leq b_2$ and $a_3 \leq b_3$.

What is the Hasse diagram?

000 100
001 101
010 110
011 111



Divisor poset of 30

looks the same

Subset poset of $\{1,2,3\}$

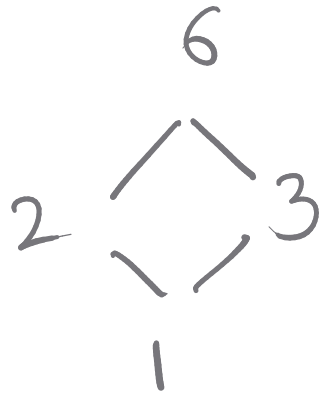
looks the same

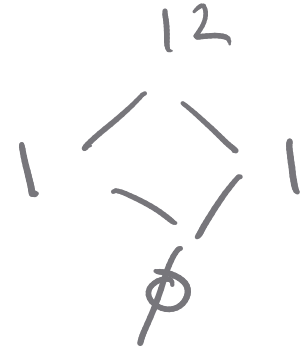
String poset of 3 bit binary strings.

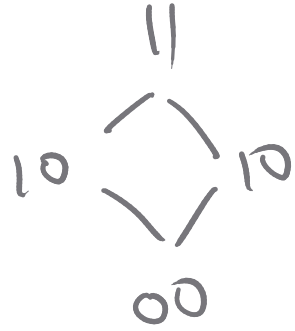
cube

Why?

Div. poset of 6 | subsets of $\{1,2\}$







Div. poset of 8