ASSIGNMENT 5 (DUE 29 AUGUST 2025)

MATH2301, SEMESTER 2, 2025

1. General remarks

(1) The assignment is due on gradescope.
(2) Please read the academic integrity policy for assignments. Remember that if you want an extension, you must ask at least 24 hours ahead of the deadline.
(3) The words "show" and "prove" are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.
(4) If you are having trouble with any of the points, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

2. Problems

2.1. **Problem.** A directed graph is called *strongly connected* if for every i and j, there is a path from vertex i to vertex j (we allow length 0 paths).

Select true or false.

(1) A graph with adjacency matrix A is strongly connected if and only if for some n, the boolean power A^{*n} has all entries equal to 1.
(2) A graph with adjacency matrix A is strongly connected if and only if for some n, the boolean sum

$$I + A + A^{*2} + \cdots + A^{*n}$$

has all entries equal to 1.

If the statement is true, you should be able to justify it. Otherwise, you should be able to give a counter-example. But you need not turn in the justification or counter-examples.

2.1.1. *Solution.*

(1) False. Consider the graph with two vertices u, v and edges $u \to v$ and $v \to u$. It is strongly connected, but no boolean power has all entries 1. The even positive powers have diagonal entries equal to 1 and the odd positive powers have off-diagonal entries equal to 1.
(2) True. Let the graph have n vertices. If the graph is strongly connected, then there is a path of length at most $n-1$ from every vertex to every vertex. So $I + A + \cdots + A^{*(n-1)}$ has all entries equal to 1.

    Conversely, if there is an m such that $I + A + \cdots + A^{*m}$ has all entries equal to 1, then there is a path of length at most m between every pair of vertices. In particular, there is a path between every pair of vertices. So the graph is strongly connected.

2.2. **Problem.** Let G be the directed graph of a relation, and let A be the boolean adjacency matrix of G. Consider the following statements.

(1) If $A = A + A^{*2}$, then the relation is transitive.
(2) If the relation is transitive, then $A = A + A^{*2}$.

Decide whether (1) or (2) or both are true. Justify your answers.

Remember that if you think a statement is false, it is enough to provide one example where it fails. If you think it is true, you must give a reasoning that works for all examples.

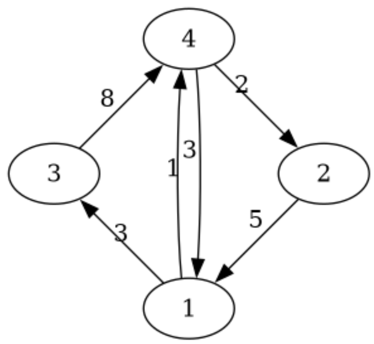
Caution: There is no subtraction in boolean arithmetic. So if $x + y = x + z$, then you cannot conclude that $y = z$.

2.2.1. *Solution.* Both statements are true.

Suppose we have $A = A + A^{*2}$. Let us prove that the relation is transitive. If $i \to j$ is an edge and $j \to k$ is an edge, then we have a length 2 path $i \to k$. This means that the (i,k) entry of $A + A^{*2}$ is 1. Since $A = A + A^{*2}$, the (i,k) entry of A is also 1, so $i \to k$ is an edge.

Suppose the relation is transitive. If an entry of A is 1, then the corresponding entry of $A + A^{*2}$ is also 1. So we need to make sure that if an entry of $A + A^{*2}$ is 1, then the corresponding entry of A is also 1. Consider the (i, j) entry of $A + A^{*2}$. If it is 1, then there is a path of length 1 or 2 from i to j. If there is a path of length 1, then $A_{i,j} = 1$. If there is a path of length 2, say $i \to k$ and $k \to j$, then by transitivity, there is actually a path of length 1 $i \to j$. So, in any case $A_{i,j} = 1$.

2.3. **Problem.** Find (using weighted adjacency matrices) the minimum cost of paths between any pair of vertices in the following graph. Assume that each vertex has loops of length 0 (not shown).



2.3.1. *Solution.* Let us add loops with weight 0 so that the n-th min/plus power of the weighted adjacency matrix will account for all paths of length up to n. Then the weighted adjacency matrix is as follows.

$$W = \begin{bmatrix} 0 & \infty & 3 & 1 \\ 5 & 0 & \infty & \infty \\ \infty & \infty & 0 & 8 \\ 3 & 2 & \infty & 0 \end{bmatrix}.$$

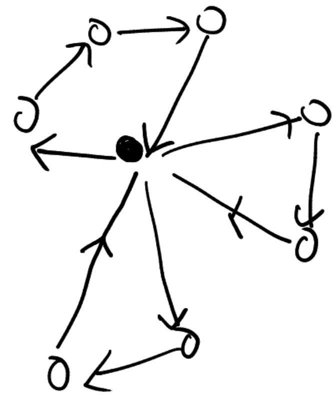
The min-plus powers are as follows (we only need up to the cube); these give the minimum-cost paths between any pair of vertices.

$$W^{\odot 2} = \begin{bmatrix} 0 & 3 & 3 & 1 \\ 5 & 0 & 8 & 6 \\ 11 & 10 & 0 & 8 \\ 3 & 2 & 6 & 0 \end{bmatrix}, \quad W^{\odot 3} = \begin{bmatrix} 0 & 3 & 3 & 1 \\ 5 & 0 & 8 & 6 \\ 11 & 10 & 0 & 8 \\ 3 & 2 & 6 & 0 \end{bmatrix}.$$

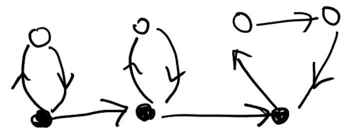
2.4. **Problem.** Let G be a graph and A its boolean adjacency matrix. Let p be the period of the eventually periodic sequence $A, A^{*2}, A^{*3}, \ldots$.

Through experiments, try to understand p for the following kinds of graphs:

Windmills/flowers: one central vertex (black) surrounded by multiple cycles of varying lengths. For example, below is a windmill with 3 cycles of length 4, 3, 3.



Trains: A backbone (black) carrying cycles of varying lengths. For example, below is a train with cycles of length 2, 2, 3.



(The black/white colouring is only for illustration, and has no other importance.)

2.4.1. *Solution.* For windmills, p is the gcd (greatest common divisor) of the cycles. For trains, p is the lcm (least common multiple) of the cycles. I will let you figure out why.