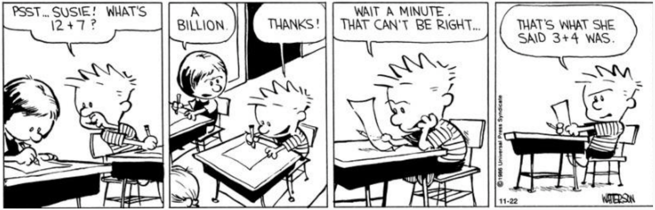
Games, graphs, and machines



August 18, 2025

A much simplified arithmetic!

- $[0]$ = The set $\{0\}$.
- $[1]$ = The set of all positive numbers.

"Boolean algebra"

pos - pos

Subtraction does not work!

$$[0] + [0] = [0] \qquad [0] \cdot [0] = [0]$$

zero + pos = pos

$$[0] + [1] = [1] \qquad [0] \cdot [1] = [0]$$

$$[1] + [1] = [1] \qquad [1] \cdot [1] = [1]$$

Old rules of arithmetic continue to hold.

$$(2 \times 3 + 5) \cdot 0 = 0$$

$$([1] \times [1] + [1]) \cdot [0] = [0]$$

Boolean powers

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find A^{*k} for $k = 1, 2, 3, \ldots$.

$\begin{pmatrix} 0 & \text{pos} \\ \text{pos} & 0 \end{pmatrix}$

Can you explain the pattern using a graph?

$* = $ Bool product.

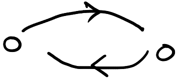
$$\begin{pmatrix} 0 & \text{pos} \\ \text{pos} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \text{pos} \\ \text{pos} & 0 \end{pmatrix} = \begin{pmatrix} \text{pos} & 0 \\ 0 & \text{pos} \end{pmatrix}$$

equ.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in Bool.}$$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A^{*2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad A^{*3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^{*4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad A^{*5} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

A = Adj matrix of  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^{*k}_{ij} = 0$ if no path from $i \to j$ len k

$= 1$ if there is.

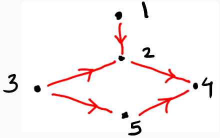
(Shadow of usual power counts paths)

$A^{*\text{even}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A^{*\text{odd}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^{*100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Existence of paths

Let A be the adjacency matrix of the graph.



$k = 100$

- Find A^{*k} for large k.
- Find $\underbrace{I + A + A^{*2} + \cdots + A^{*k}}_{\text{what does this represent?}}$ for large k.

$$A^{*100} = 0$$

$$A^{*3} = A^{*4} = \cdots \cdots = 0$$

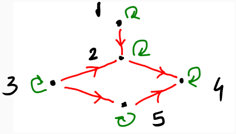
(NO length 3, 4, 5 .. paths)

$$\left(I + A + A^{*2} + \cdots + A^{*k}\right)_{ij} = \text{Existence of path } i \to j \text{ of length} \le k$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ & & & & \end{pmatrix}$$

Adding loops

Suppose we add loops.



- Now what is A^{*k} for large k?
- What about $\underbrace{I + A + A^{*2} + \cdots + A^{*k}}_{B_k}$?

$$A^{*100} = \begin{pmatrix} 1 & \\ 1 & \end{pmatrix} = B_{100} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exactly 100

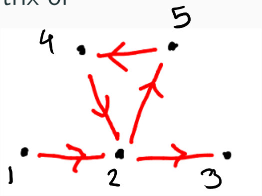
||

Up to 100 (incl

Accum power of adj matrix of graph without loops.

Modular arithmetic?

Let A be the adjacency matrix of



len

12 : 1, 4, 7, 10, ...

13 : 2, 5, 8, 11,

14 : 3, 6, 9, 12, ...

15 : 2, 5, 8, 11, ...

Can you describe A^{*k} for all k?

A^1 : 12, 23, 25, 42, 54

A^2 : 15, 13, 24, 43, 45, 52

A^3 : 14, 22, 44, 55, 53

A^4 : 12, 23, 25, 42, 54

A^5 : 1

A^6 :

seems to be 3-periodic.

Eventually periodic?

Let A be the boolean adjacency matrix of a graph.

Claim — The sequence of matrices $A, A^{*2}, A^{*3}, \ldots$ is ~~eventually~~ periodic.

True or false?