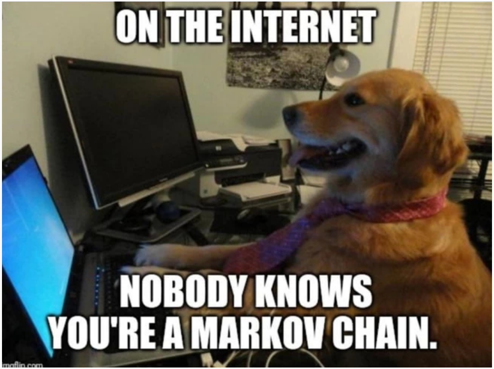
Games, graphs, and machines



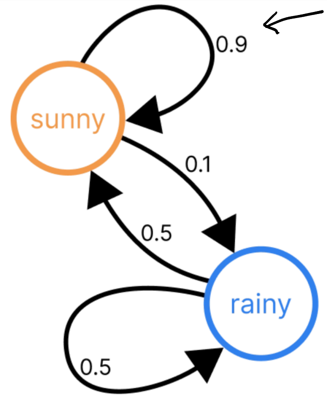
Graphs & matrices to analyse random processes.

August 25, 2025

Warm up

Write the transition matrix A for the following Markov chain.

1



2

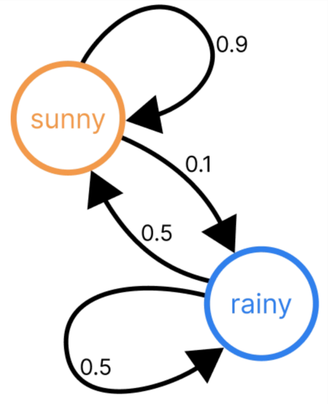
Probability.

$$\begin{array}{ccc} & 1 \quad\; 2 & \\ \begin{array}{c}1\\2\end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} & \begin{array}{c}\to 1\\ \to 1\end{array} \end{array}$$

Warm up

Write the transition matrix A for the following Markov chain.

Calculate A^2. What do entries of A^2 represent?



$$A = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.86 & 0.14 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

A^3 = 3-day forecast

A^n = forecast after n-days.

Why does kth power represent k-step probabilities?

2x2

$$A^2_{i,j} = \underbrace{A_{i,1} \cdot A_{1,j}} + \underbrace{A_{i,2} \cdot A_{2,j}}.$$

$i \to 1 \to j \; + \; i \to 2 \to j$

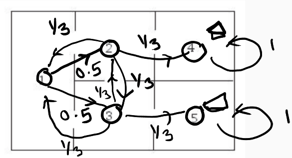
$\cdot$ = Prob of "and"

$+$ = Prob. of "or"

A maze

A maze used for training rats has the following shape.

* except for cheese.



Suppose that at every stage, the rat picks a door at random and goes through that door.* Write the Markov chain and the transition matrix.

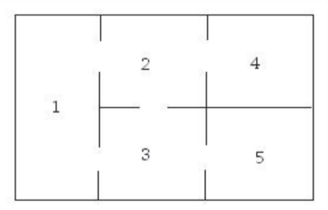
$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Powers of the maze

See if the powers of the matrix converge and interpret the result.

$\longrightarrow$ end up.

$$\lim_{n\to\infty} A^n = \begin{array}{c} 1 \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 & 0 & 0 & 50\% & 50\% \\ 0 & 0 & 0 & 62\% & 38\% \\ 0 & 0 & 0 & 38\% & 62\% \\ 0 & & & 100\% & \\ 0 & & & & 100\% \end{pmatrix}$$

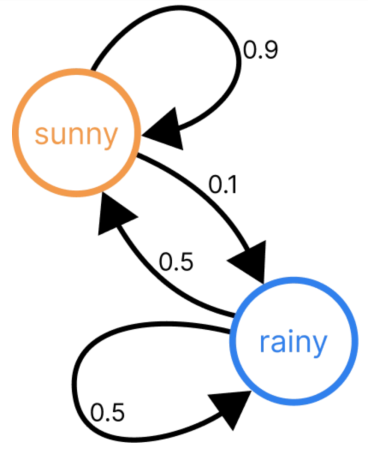


Start

$\longrightarrow$ Long term probabilities

Powers of the weather

See if the powers of the matrix converge and interpret the result.



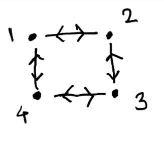
$$\lim_{n\to\infty} A^n = \begin{pmatrix} 83\% & 17\% \\ 83\% & 17\% \end{pmatrix}$$

A random walk

Let G be a directed graph. A random walk on G consists of walking on G by the following simple rule: pick an outgoing edge at random and follow it.

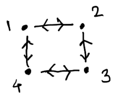
Write the transition matrix for the random walk on

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ - & - & - & - \\ & - & & \end{pmatrix} \end{array}$$



Powers of the random walk

See if the powers of the matrix converge and interpret the result.



Don't converge.

Oscillate!

1. convergence, but outcomes depend on starting state.
2. —||— outcomes indep.
3. not conv. oscillation.

Understanding large powers

It turns out that the sunny/rainy matrix A can be written as

$$A = EDE^{-1},$$

where $E = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}.$

Use this to understand A^k for large k.

When do large powers converge?

Suppose $A = EDE^{-1}$, where D is diagonal. When will A^k converge (to a matrix with finite entries) as k grows?