Games, graphs, and machines

https://miachenmtl.github.io/random-jumps/

August 26, 2025

Three possibilities

Let A be the transition matrix of a Markov chain.

1. $\lim_{n\to\infty} A^n$ exists and has the same rows. ← eventual behaviour is indep of initial state
2. $\lim_{n\to\infty} A^n$ exists but has different rows.
3. $\lim_{n\to\infty} A^n$ does not exist.

Oscillation

Understanding large powers

Eigenvalues.

It turns out that the sunny/rainy matrix A can be written as

$$A = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$$

"digonalisation"

$$A = EDE^{-1},$$

where $E = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$ $\quad E^{-1} = \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}$.

"eigenvalues of A"

Use this to understand A^k for large k.

$$E \cdot E^{-1} = I$$

$$E^{-1} \cdot E = I$$

$$A^2 = ED\underbrace{E^{-1}E}D E^{-1} = ED^2E^{-1}$$

$$A^3 = ED^3E^{-1} \qquad A^4 = ED^4E^{-1} \qquad A^n = ED^nE^{-1}$$

$$D^2 = \begin{pmatrix} 1^2 & 0 \\ 0 & 0.4^2 \end{pmatrix} \qquad D^3 = \begin{pmatrix} 1^3 & \\ & 0.4^3 \end{pmatrix} \qquad D^n = \begin{pmatrix} 1^n & \\ & 0.4^n \end{pmatrix}$$

$$\lim_{n \to \infty} D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\lim A^n = E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} E^{-1}$$

$$\begin{pmatrix} 5/6 & 1/6 \\ 5/6 & 1/6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}$$

When do large powers converge?

Suppose $A = EDE^{-1}$, where D is diagonal. When will A^k converge (to a matrix with finite entries) as k grows? → depends on eigenvalues.

Ex. $D = \begin{pmatrix} 1 & \\ & 0.4 \end{pmatrix}$ ✓

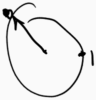
2 periodic oscillation. ← $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$ → blow-up (entries go to ∞) → can't happen in Markov chain.

$\begin{pmatrix} 3 & \\ & 4 \end{pmatrix}$ → blowup

$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ → 2-periodic.

ζ

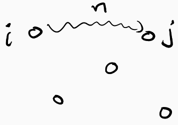


→ $D = \begin{pmatrix} 1 & \\ & \zeta \end{pmatrix}$ $\quad \zeta^3 = 1$

The Perron-Frobenius theorem

Cond

Theorem
Let A be the transition matrix of a Markov chain. Suppose there exists an n such that for every i and j, there is a path of length n from state i to state j. Then



1. $\lim_{n\to\infty} A^n$ *exists.*
2. *The limiting matrix has identical rows.*
3. *The limiting row vector v is the unique vector whose entries sum to 1 and which satisfies the equation*

$$\boxed{vA = v.}$$

$$\begin{pmatrix} - v - \\ - v - \\ - v - \\ - v - \end{pmatrix}$$

Does it apply?

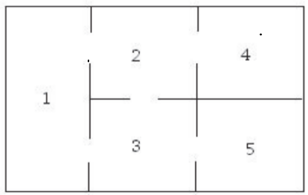
To the random walk of the rats?

- with cheese in rooms 4/5? NO.
- without? YES $\therefore$ $n=10$

Any n

$i=4$

$j=1$



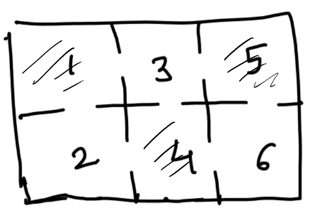
There is an n s.t. path of length n between any two states.

A slightly different maze

n = even

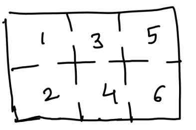
↳ doesn't work

n = odd



PFT does not apply.

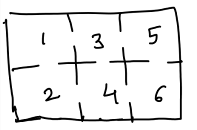
A slightly different maze




```
A = matrix([[0, 1/2, 1/2, 0, 0,0],
            [1/2,0,0,1/2,0,0],
            [1/3,0,0,1/3,1/3,0],
            [0,1/3,1/3,0,0,1/3],
            [0,0,1/2,0,0,1/2],
            [0,0,0,1/2,1/2,0]])
(A^(100)).n(10)
```

```
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
```

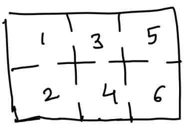
A slightly different maze




```
(A^(101)).n(10)

[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

A slightly different maze




```
(A^(101)).n(10)
```

```
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

```
A.eigenvalues()
```

```
[1, 1/2, 1/6, -1/6, -1/2, -1]
```

✓

do not matter

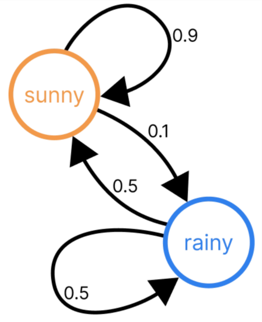
responsible for oscillation

Does it apply?

To the weather forecaster?

$n = 1 \quad$ PFT ✓

$\lim_{n \to \infty} A^n$ exists has same rows



$$vA = v \qquad \begin{aligned} x+y&=1 \\ y&=1-x \end{aligned}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$x(0.9) + (1-x)\,0.5 = x$$
$$x\,0.1 + (1-x)\,0.5 = 1-x$$

$$x = 5/6 \qquad y = 1/6$$

$$\begin{pmatrix} 5/6 & 1/6 \\ 5/6 & 1/6 \end{pmatrix}$$

Why does does Perron-Frobenius hold?

The crucial point is that under the hypotheses:

- only one eigenvalue is 1
- all the others have absolute value < 1