

**WEEK 12 WORKSHOP**  
**MATH2301, SEMESTER 2, 2025**

1. GRUNDY LABELS

The goal of this worksheet is to practice computing Grundy labels and using them to find winning moves. Find the Grundy value of Chomp-the-graph for the graph

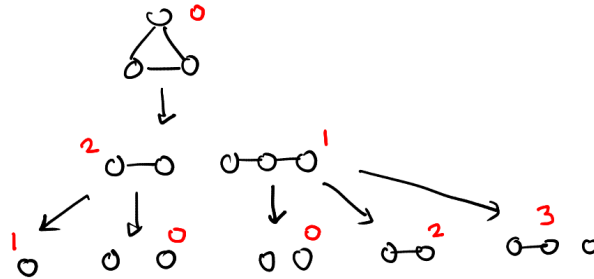


Using the Grundy value, find all possible winning moves for the first player.

A reminder of the rules: two players take turns removing either a vertex or an edge of  $G$ . When a vertex is removed, all edges adjacent to it are automatically removed. When an edge is removed, no other vertices (or edges) are removed. As usual, the player who cannot make a move loses.

For practice, do the problem with different graphs.

**1.1. Solution.** The game is a sum of chomp-the-graph for the triangle and a singleton vertex. So its Grundy label is the xor-sum of the Grundy label for chomp-the-graph starting with a triangle and chomp-the-graph starting with a single vertex. Here is the following game tree for the triangle with the Grundy labels computed using the mex rule.



So the Grundy label of chomp-the-graph for triangle + vertex is  $0 \oplus 1 = 1$ .

A winning move consists of (if possible) moving the vertex to a state with label  $1 \oplus 1$  or the triangle to a state with label  $0 \oplus 1$ . Both are possible: either delete the vertex or delete an edge of the triangle. Here  $\oplus$  stands for xor-sum.

2. STABLE EQUIVALENCE

True or false:

- (1)  $\text{Nim}(2, 3) \sim \text{Nim}(1, 4)$ .
- (2) If  $A \sim B$ , then  $A + C \sim B + C$  for any game  $C$ .
- (3) If  $A + C \sim B + C$ , then  $A \sim B$ .
- (4) If  $A \sim B$ , then the number of winning first moves in  $A$  is the same as the number of winning first moves in  $B$ .

2.1. **Solution.** By the Sprague-Grundy theorem, two games are equivalent if and only if they have the same Grundy value.

- (1) The Grundy value of Nim(2,3) is  $2 \oplus 3 = 1$ . The Grundy value of Nim(1,5) is  $1 \oplus 4 = 5$ . So the two games are not equivalent.
- (2) True.
- (3) True.
- (4) False. For example, chomp-the-graph in the first problem is equivalent to Nim(1). It had 2 winning moves, but Nim(1) has only one.

### 3. MISERE PLAY (ONLY FOR FUN; NOT ON THE EXAM)

In *Misere play*, the player who makes the last move loses (so the player who cannot make a move wins).

- (1) Try playing *nim* with the misere rule.
- (2) How would you modify the rules of strategic ( $N/P$ ) labelling for Misere play?
- (3) Let us assign misere Grundy labels by the following rules: label the terminal nodes as 1; for non-terminal nodes, use the same rule: a node gets the mex of the labels of the children. With this labelling, convince yourself that a non-zero label means an  $N$  state and a zero label means a  $P$  state. But it is not true that the label of  $G + H$  is the xor-sum of the labels of  $G$  and  $H$ . Find an example where this fails.

3.1. **Solution.**

- (1) DIY
- (2) Terminal states are  $N$ . The other rules are the same.
- (3) A 0 state can only lead to a non-zero state. A non-zero state must have a child that is 0. Misere Nim(1) is 0, but misere Nim(1,1) is 1, which is not  $0 \oplus 0$ .

### 4. STRATEGIC LABELLING FOR PARTISAN GAMES (ALSO ONLY FOR FUN)

In partisan games, there are 4 possible outcomes: L (left has a winning strategy, irrespective of whether they go first or second), R (right has a winning strategy), N (first player has a winning strategy, irrespective of whether they are R or L), P (second player has a winning strategy). This means that we can label every game state as L, R, N, or P. Given a state, its R-children are the states reachable by R moves and L-children are the states reachable by L moves. Using R and L children, find the rules for strategic labelling. Here is start.

- (1) If a state has no children (R or L), then label it P.
- (2) If a state has R-children but no L-children, then label it R.
- (3) If a state has L-children but no R-children, then label it L.
- (4) If a state has an R-child labelled P/R and an L-child labelled P/L, then label it N.
- (5) ... more ...

For impartial games, we saw that adding a P game does not change the outcome. Is this still true for partisan games?

4.1. **Solution.** Here are more rules. I will let you figure out why they work.

- (5) If a state has an R-child labelled P/R but no L-child labelled P/L, then label it R.
- (6) If a state has an L-child label P/L but no R-child labelled P/R, then label it L.
- (7) If a state has no R-child labelled P/R and no L-child labelled P/L, then label it P.

I think they cover all possibilities (let me know if I missed something).

It is still true that adding a P game does not change the outcome.