WEEK 5 WORKSHOP

MATH2301, SEMESTER 2, 2025

1. BOOLEAN ARITHMETIC WITH NEGATIVE NUMBERS

Let us see what happens with boolean arithmetic if we allow negative numbers.

Consider the equivalence relation on **R** defined by $a \sim b$ if both a and b are zero, or if both a and b have the same sign.

1.1. **Problem.** What are the equivalence classes?

Solution. There are three equivalence classes: $[0]$, $[1]$, and $[-1]$.

1.2. **Problem.** Try to define $+$ and $\cdot$ on the equivalence classes by following the same procedure as before. To compute $A+B$ for two equivalence classes A and B, we pick a number in A, a number in B, add them, and then take the equivalence class of the sum. Similarly for $\cdot$. Does this procedure define a consistent $+$ or $\cdot$ on the equivalence classes?

Solution. Consistency fails for $+$ but works for $\cdot$.

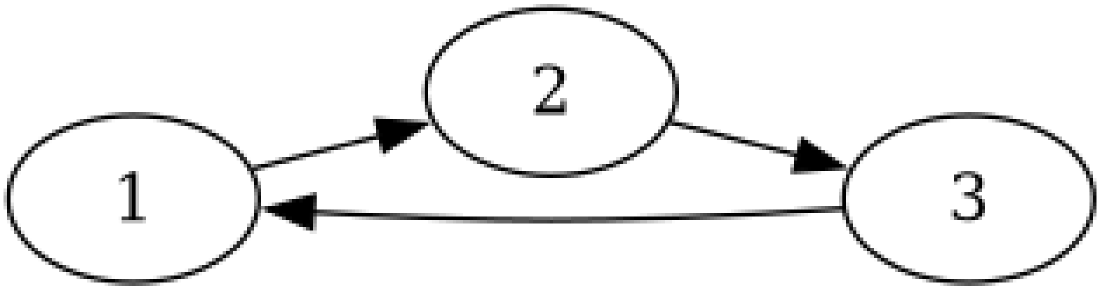
For $+$, the issue is that the result of $[1]+[-1]$ depends on the chosen representatives. It could be $[1]$, or $[0]$, or $[-1]$. Said simply, positive plus negative could turn out to be either positive, or zero, or negative.

Multiplication works without any problem.

2. OSCILLATION VERSUS STABILISATION

2.1. **Problem.** Construct a boolean matrix whose boolean powers oscillate with a cycle of 3.

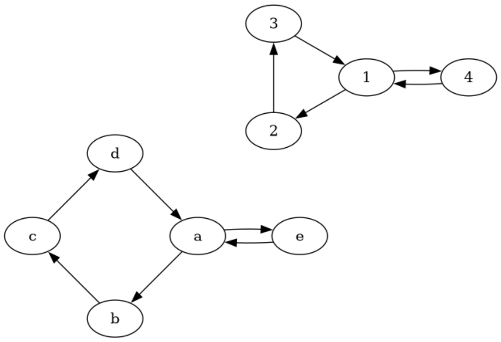
Solution. The adjacency matrix of a 3 cycle does the trick. I will let you figure out why.



2.2. **Problem.** We saw in the lecture that the powers of any boolean matrix A are eventually periodic.

What is the period in the following two examples?

Date: August 18, 2025.



Solution. In the first example, the period is 1. In the second example, the period is 2.

I will leave it to you to figure out why. Can you conjecture how the period depends on the graph?

2.3. **Problem.** Let A be any boolean matrix. True or false: the accumulated powers $I + A + A^2 + A^3 + \cdots + A^n$ stabilise. That is, after a certain n, the value of $I + A + \cdots + A^n$ does not depend on n.

Solution. True. Look at a particular entry, say (i, j) If there an n such that $A^n_{i,j} = 1$, let $N_{i,j}$ be the smallest such n. Then, for all $n \geqslant N_{i,j}$, the (i, j)-entry of the accumulated product is 1. If there is no n such that $A^n_{i,j} = 1$, then the (i, j)-entry of the accumulated product is always 0. Take N to be the maximum of the $N_{i,j}$. Then, for $n \geqslant N$, the accumulated power does not change.

There is also a graph theoretic explanation. Suppose we have a graph with n vertices. If there is a path from a vertex i to a vertex j. Then there must be a path of from i to j that has length at most n (why? – if your path is longer, it must have a cycle; eliminate it to get a shorter path). This means that the accumulated power does not change after n.

3. POWERS VERSUS ACCUMULATED POWERS

3.1. **Problem.** Convince yourselves, that in boolean algebra, we have

$$(I + A)^n = I + A + A^2 + \cdots + A^n.$$

Start by taking $n = 1, 2, 3, \ldots$. It may be helpful to notice that $A + A = A$.

Solution. We have $A + \cdots + A = A$. So $(I + A)^2 = I + 2A + A^2 = I + A + A^2$. Then $(I + A)^3 = (I + A)(1 + A + A^2) = I + A + A^2 + A^3$, and so on. You just have to keep track of whether a power of A appears in the product or not. Even if it appears multiple times, it has the same effect as appearing once.

3.2. **Problem.** Let A be the adjacency matrix of a graph. Let B be the adjacency matrix of the same graph with all self-loops added. Explain the phenomenon: the n-th power of B is the n-th accumulated power of A.

That is,

$$B^n = I + A + \cdots + A^n.$$

Solution. Observe that $B = I + A$. Raising to n on both sides and using the previous problem gives

$$B^n = I + \cdots + A^n.$$