Games, graphs, and machines

In tropical algebra, the sum of two numbers is their minimum and the product of two numbers is their sum.

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Periodic?

Let A be the boolean adjacency matrix of a graph.

Claim — The sequence of matrices $A, A^{*2}, A^{*3}, \ldots$ is periodic.

True or false?

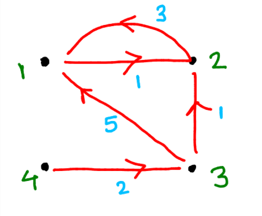
Warm up with min/plus arithmetic

Remember that $\oplus = \min$ and $\odot = +$. Find

$$2 \odot (3 \oplus 1) \oplus 3 \odot (\infty \oplus 2).$$

Weighted adjacency matrix

Write the min/plus weighted adjacency matrix of the graph. Assume that the loops have weight 0 (not shown).



Min/plus powers

Find the min/plus square and cube of the adjacency matrix. What do its entries represent?

Why do min/plus powers give shortest paths?

For example, the third power:

$$\begin{aligned} A^3_{i,j} &= (A^2_{i,1} \odot A_{1,j}) \oplus (A^2_{i,2} \odot A_{2,j}) \oplus (A^3_{i,3} \odot A_{3,j}) \oplus (A^2_{i,4} \odot A_{4,j}) \\ &= \min(A^2_{i,1} + A_{1,j}, A^2_{i,2} + A_{2,j}, A^3_{i,3} + A_{3,j}, A^2_{i,4} + A_{4,j}) \end{aligned}$$

When do we stop?

Assume:

1. we have all loops with weight 0,
2. all weights are non-negative.

Theorem
Let n be the number of vertices. Then $A^{\odot(n-1)} = A^{\odot n} = A^{\odot(n+1)} = \cdots$.