ASSIGNMENT 1 (DUE 1 AUGUST 2025)

MATH2301, SEMESTER 2, 2025

1. General remarks

(1) The assignment is due on gradescope.

(2) Please read the academic integrity policy for assignments.

(3) To prove that a property holds for all objects under consideration, you must give an argument that works for all objects. Giving a few examples where the property holds is usually not enough.

(4) To disprove that a property holds for all objects under consideration, it is enough to give *one example* where the property does not hold.

(5) The words "show" and "prove" are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.

(6) If you are having trouble with any of the points mentioned above, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

2. Problems

2.1. **Problem.** Let $S = \mathbf{R} - \{0\}$. Define $R \subset S \times S$ as

$$R = \{(x, y) \mid x^2 y = 12.\}$$

Then R is the input/output (I/O) relation of a function f. Find $f(1)$ and $f(2)$.

2.1.1. *Solution.* $f(1)$ is the unique value such that $(1, f(1)) \in R$. For $(1, f(1))$ to be in R, we must have $1^2 \cdot f(1) = 12$. This gives $f(1) = 12$.

Similarly, we get $f(2) = 3$.

In general, we have $f(x) = 12/x^2$.

2.2. **Problem.** Let R and T both be relations on a set S. Decide if the following statements are true or false, and justify your answer.

(1) If R and T are symmetric, then $R \cup T$ is symmetric.

(2) If R and T are transitive, then $R \cup T$ is transitive.

2.2.1. *Solution.*

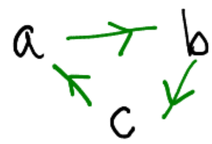
(1) This is false. For example, we can have $R = \{(1,2)\}$ and $T = \{(2,3)\}$. Both R and T ar trivially transitive because they each only have one element. But $R \cup T$ is not transitive, because it does not contain $\{(1,3)\}$.

(2) This is true. If $(a, b) \in R \cup T$ then either $(a, b) \in R$ or $(a, b) \in T$. Suppose $(a, b) \in R$. Then since R is symmetric, we have $(b, a) \in R$, and so $(b, a) \in R \cup T$. By a similar argument, if $(a, b) \in T$ then $(b, a) \in T$ and so $(b, a) \in R \cup T$.

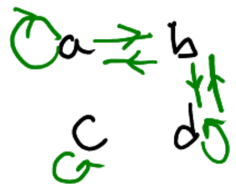
2.3. **Problem.** Consider the following graphs. For each one, write down which of the following properties are satisfied by the relation represented by the graph: reflexivity, symmetry, transitivity, being I/O of a function. You do not have to justify your answers, but you should think about the justifications instead of guessing.

Date: August 1, 2025.

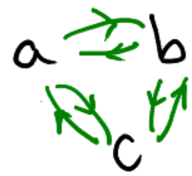
2.3.1. *Graph 1.*



2.3.2. *Graph 2.*



2.3.3. *Graph 3.*



2.3.4. *Solution.*

Graph	Reflexive	Symmetric	Transitive	I/O
1	No	No	No	Yes
2	No	Yes	No	No
3	No	Yes	No	No

2.4. **Problem.** Let $S = \mathbf{R} \times \mathbf{R}$. Define a relation R on S as follows:

$$R = \{((a,b),(c,d)) \mid a + d = b + c\}.$$

(1) Prove that R is an equivalence relation.

(2) Describe the equivalence classes in words, and draw sketches in $\mathbf{R}^2$, of the equivalence class of $(1,2)$ and of $(0,0)$.

2.4.1. *Solution.* Let us prove reflexivity, symmetry, and transitivity.

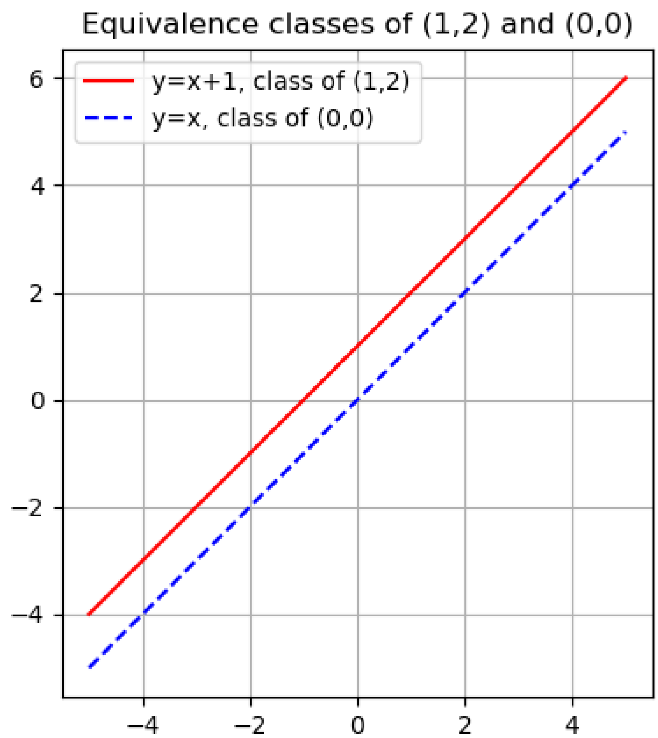
- It is reflexive. Note that $a + b = b + a$, so $((a,b),(a,b)) \in R$ for each element $(a,b) \in S$.
- It is symmetric. If $((a,b),(c,d)) \in R$ then $a + d = b + c$. We can rewrite this as $c + b = a + d$, which means that $((c,d),(a,b)) \in R$. This is true for all elements $a,b,c,d \in \mathbf{N}$.
- It is transitive. Suppose that $((a,b),(c,d))$ and $((c,d),(e,f))$ are both in R. We know that $a+d = b+c$ and that $c + f = d + e$. Adding the two equations, we see that $a + d + c + f = b + c + d + e$. We can now cancel $c + d$ from both sides to see that $a + f = b + e$, and so $((a,b),(e,f)) \in R$.

For the picture, note that the equivalence class of a fixed pair (a,b) is the set of all (x,y) such that

$$a + y = b + x,$$

which is the same as all (x,y) that satisfy $x - y = a - b$ or $y = x + (b - a)$. This describes a straight line with slope 1 and y-intercept $b - a$.

In particular, the equivalence classes of $(1,2)$ and $(0,0)$ look like this.


Equivalence classes of (1,2) and (0,0)
y=x+1, class of (1,2)
y=x, class of (0,0)
6
4
2
0
−2
−4
−4
−2
0
2
4