WEEK 6 WORKSHOP

MATH2301, SEMESTER 2, 2025

1. The Perron–Frobenius property

Let G denote a Markov chain with transition matrix A.

1.1. **Problem.** When does the Perron–Frobenius theorem (PFT) apply? Give your answer in terms of

(1) the existence/non-existence of paths,
(2) powers of A,
(3) powers of the boolean adjacency matrix B.

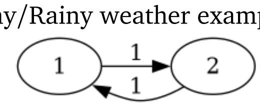
Solution.

(1) There exists an n such that there is a path of length n between any pair of vertices.
(2) There exists an n such that all entries of A^n are non-zero.
(3) There exists an n such that all entries of $B^{\cdot n}$ are 1.

1.2. **Problem.** Construct examples of graphs G (omit the probabilities) where

(1) PFT applies,
(2) PFT does not apply,
(3) the graph is strongly connected and yet PFT does not apply.

Solution.

(1) Many examples. For example, the Sunny/Rainy weather example from class.
(2) 

(3) See above.

2. The GCD condition

Recall that a convenient way to verify that PFT applies is:

- make sure the graph is strongly connected,
- find a vertex v and directed cycles based at v of length a and b with $\gcd(a, b) = 1$.

2.1. **Problem.** Let G be the Markov chain of snakes and ladders on a 10×10 board where we continue from 0 if we go beyond 100. (So rolling a 4 at 98 takes us to 2). Verify the two conditions above.

Solution. Taking jumps of 1 takes us from any square to any other square. So the graph is strongly connected. Taking 100 jumps of 1 gives a cycle from 0 to 0. Taking 98 jumps of 1 and 1 jump of 2 gives another cycle from 0 to 0. The lengths of these cycles are 100 and 99, respectively, whose gcd is 1.

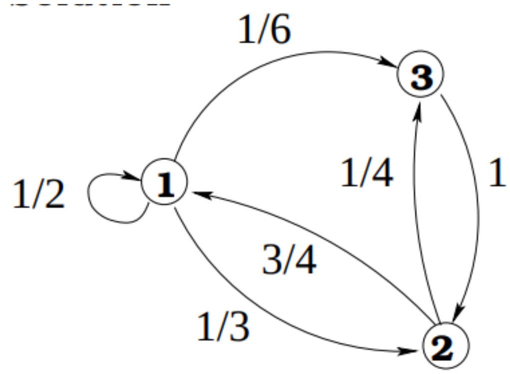
2.2. **Problem.** Does your argument work if we change the size of the board? What if we change the 6-sided die to an 8-sided die?

Solution. Nothing in the argument essentially changes.

Date: August 25, 2025.

3. COMPUTING THE STEADY STATE

3.1. **Problem.** Let A be the transition matrix of the following Markov chain. Use PFT to find $\lim_{n\to\infty} A^n$.



Solution. The transition matrix is

$$A = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}.$$

Check that PFT applies. So all the rows of $\lim A^n$ are identical and equal to $[x, y, z]$ satisfying

$$x + y + z = 1,$$

and $[x, y, z]A = [x, y, z]$, which translates into

$$1/2 \cdot x + 3/4 \cdot y = x, \quad 1/3 \cdot x + z = y, \quad 1/6 \cdot x + 1/4 \cdot y = z.$$

We substitute $z = 1 - x - y$, to get

$$1/2 \cdot x + 3/4 \cdot y = x, \quad 1/3 \cdot x + 1 - x - y = y, \quad 1/6 \cdot x + 1/4 \cdot y = 1 - x - y,$$

which simplifies to

$$3/4 \cdot y = 1/2 \cdot x, \quad 2/3 \cdot x + 2y = 1, \quad 7/6 \cdot x + 5/4 \cdot y = 1.$$

Substituting $x = 3/2y$, we get

$$3y = 1, \quad 3y = 1.$$

So $y = 1/3$ and $x = 1/2$ and $z = 1/6$. That is,

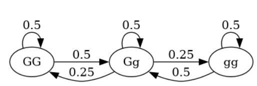
$$\lim_{n\to\infty} A^n = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}.$$

(You can probably solve the system of linear equations in a more efficient way.)

3.2. **Menedlian genetics.** Gregor Mendel observed a particular gene in garden peas that exists in two types: G or g. Each pea plant has a pair of genes. So a plant could be of type GG, or $Gg = gG$, or gg. We take a plant and fertilise it with a plant of type Gg to produce an offspring. We do the same process starting with the offspring (always mating with a Gg plant), and continue. The offspring inherits one gene from each parent with equal probability. So, for example, if we fertilise GG with Gg, then the offspring will be GG with probability $1/2$ and Gg with probability $1/2$. If we fertilise Gg with Gg, the offspring will be GG with probability $1/4$, $Gg = gG$ with probability $1/2$, and gg with probability $1/4$.

Problem. Taking $\{GG, Gg, gg\}$ as the vertex set, describe the Markov chain.

- Solution



Problem. Decide if PFT applies.

- Solution

Problem. Write the transition matrix A and compute the first few powers. Observe the row corresponding to Gg. What do you notice? Can you interpret your result? What about the other two rows?

- Solution We have the following transition matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

  Its second power is $\begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{pmatrix}$

  Its third power is $\begin{pmatrix} \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{2} & \frac{5}{16} \end{pmatrix}$

  It seems like the second row stays the same! This means that, if we start with the Gg strain, then at any stage, we have $1/4$ probability of getting GG, $1/2$ of getting Gg, and $1/4$ of getting gg. The first and the third rows do not stay constant.

4. Random walks (if time permits)

Let G be a directed graph. Let A be the transition matrix of the random walk on G.

4.1. **Problem.** Suppose G is symmetric, connected, and has a loop at every vertex. Show that PFT applies. Can you find the limiting distribution?

Solution. Because G is connected and symmetric, it is strongly connected. Since there is a loop at every vertex, the gcd condition is trivially satisfied.

By doing a few examples, you can guess the limiting distribution. Once you have the correct guess, it should be easy to justify (remember: the limiting distribution is the unique one satisfying certain equations. So if you show that your guess satisfies those equations you are done!)

The limiting distribution is the vector of degrees (scaled so that the sum of the entries is 1). In the degree computation, the loop contributes 1.

For example, for the graph $1-2-3$ (with the loops), the degree sequence is $(2, 3, 2)$. So the limiting distribution is $(2/7, 3/7, 2/7)$.

4.2. **Problem.** Try relaxing the conditions above and explore what happens (use a computer).

Solution. Left to you!

5. The GCD condition (if time permits)

The reason the GCD condion works is the following theorem.

Theorem: Let a, b be positive integers with $\gcd(a, b) = 1$. Then any $n > ab$ can be written as a sum of a's and b's.

(1) Try to prove it for $a = 3$ and $b = 4$.
(2) Try with $a = 3$ and $b = 5$.
(3) Try with any a and $b = a + 1$.
(4) For the general case, let r be the remainder when n is divided by a.
   - If $r = 0$, we can simply write $n = a + a + \cdots + a$ (no b's needed).
   - If $r \equiv b \pmod{a}$, what would you do?
   - If $r \equiv 2b \pmod{a}$, what would you do?
   - If $r \equiv 3b \pmod{a}$, what would you do?
   - Can you generalise?

Solution. Let us sketch the general argument. The key point is that if $\gcd(a, b) = 1$, then the numbers $0, b, 2b, \cdots, (a-1)b$ are all distinct modulo a. Therefore, in some order, these are $0, 1, \ldots, (a-1)$ modulo a. So, by subtracting an appropriate ib from n (for some i in $\{0, \ldots, a-1\}$) we get $n - ib$ that is divisible by a, say $n - ib = ja$. If $n > ab$, then $n - ib > 0$. Then we have $n = ib + ja$.