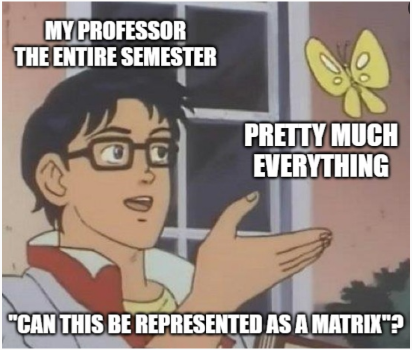
Games, graphs, and machines



August 12, 2025

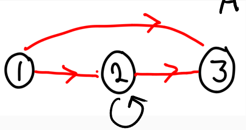
Warm up

Find the adjacency matrix A.

$$A' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB \neq BA$$

$$(AB)C = A(BC)$$



Calculate A^2 and A^3.

row i col j

$$A \cdot A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A \cdot A^2 = A^2 \cdot A$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A(A \cdot A)$$

$$(A \cdot A) \cdot A$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

row1 ← source

col3 ← target

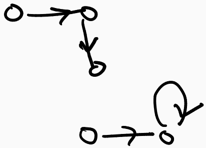
A^3 ↙ length

$$\begin{pmatrix} & & \square \\ & & \\ & & \end{pmatrix}$$

Paths	prod. matrix
1→3→3→3	3
1→2→3→3	
1→2→2→3	

Theorem
The (i, j) entry if A^k is the number of paths from vertex i to vertex j of length k.

$A^2 \leftarrow$ Length 2 paths



$K \leftarrow$ length

$A^K_{i,j}$

$i \rightarrow$ source

$j \rightarrow$ target

Why does A^k count length k paths?

Theorem
The (i,j) entry if A^k is the number of paths from vertex i to vertex j of length k.

$k = 1$ Yes ← length 1 path = edge

$k = 2$ Let us see.

len 2 from i to j →

$$A^2_{i,j} = \overbrace{A_{i,1} \cdot A_{1,j}}^{\text{Len 2 paths whose penult stop is 1}} + \underbrace{A_{i,2} \cdot A_{2,j}}_{\text{len 2 paths i to j whose penult stop is 2}} + \overbrace{A_{i,3} \cdot A_{3,j}}$$

$i \to 1$ $1 \to j$

Row i: $A_{i,1}$ A_{i2} A_{i3}

Col j: A_{1j} A_{2j} A_{3j}

Why does A^k count length k paths?

- $k = 3$

Len 3 paths from i to j whose penult stop is 1

$$A^3_{i,j} = A^2_{i,1} \cdot A_{1,j} + \underbrace{A^2_{i,2} \cdot A_{2,j}}_{\text{len3 penult stop 2}} + \underbrace{A^2_{i,3} \cdot A_{3,j}}_{\text{len3 penult is 3}}$$

✓

$i \rightsquigarrow 1$ $1 \to j$

Row i:

A^2_{i1} A^2_{i2} A^2_{i3}

Col j A

A_{1j}

A_{2j}

A_{3j}

Why does A^k count length k paths?

- $k = 4$ ✓

↙ len 4 paths partitioned by penult stop.

$$A^4_{i,j} = \underbrace{A^3_{i,1}}_{\substack{\text{Len4 paths from } i \text{ to } j \\ \text{whose penult stop} \\ \text{is } 1}} \cdot A_{1,j} + A^3_{i,2} \cdot A_{2,j} + A^3_{i,3} \cdot A_{3,j}$$

$$A^4_{i,j} = \underbrace{A^2_{i,1} \cdot A^2_{1j}} + A^2_{i2} \cdot A^2_{2j} + A^2_{i3} A^2_{3j}$$

$i \overset{2}{\rightsquigarrow} 1 \overset{2}{\rightsquigarrow} j$ ↙ len 4 path i to j whose middle stop is 1

Sum of powers

What do the entries of $A + A^2 + A^3 + A^4$ represent?

(i,j) of $(A + A^2 + A^3 + A^4)$

$$= \underbrace{A^1_{ij} + (A^2)_{ij} + (A^3)_{ij} + (A^4)_{ij}}$$

paths of length 1 or 2 or 3 or 4 from i to j.

$$A^0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Acyclic graphs

We say that G is *acyclic* if it has no (directed) cycle.

Suppose G is acyclic and has 100 vertices. What can you say about A^{100}?

Longest path

Let G be a graph with adjacency matrix A.

Using A, how will you find the length of the longest possible path in G?