

# Games, graphs, and machines

Equivalence relations

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July 28, 2025

# Equivalence relation (reminder)

$R$  is an equivalence relation on a set  $S$  if:

**Reflexive** For all  $a \in S$ , we have  $(a, a) \in R$ .

**Symmetric** If  $(a, b) \in R$ , then  $(b, a) \in R$ .

**Transitive** If  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R$ .

Ex.

=

"the same"



# Equivalence relation or not?

Is  $\sim$  an equivalence relation on  $S$ ?

1.  $S = \mathbb{R}$  and  $a \sim b$  if  $|a - b| < 1$ .

✓ Refl

Not transitive

✓ Symm

2.  $S =$  States of a chess-board and  $a \sim b$  if we can reach  $b$  from  $a$  by a sequence of legal moves.

✓ transitive

✓ refl.

X symmetric

3.

$a \sim b$  if  $ab > 0$

$S = \mathbb{R} - \{0\}$

( allow

empty seq of moves.

# Equivalence relation or not?

Is  $\sim$  an equivalence relation on  $S$ ? ( $T$  a finite set.)

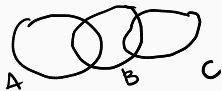
✓ 1.  $S = \text{Pow}(\overset{T}{\bullet})$  and  $A \sim B$  if  $|A| = |B|$ .

✗ 2.  $S = \text{Pow}(\overset{T}{\bullet})$  and  $A \sim B$  if  $A \subset B$ .



3.  $S = \text{Pow}(\overset{T}{\bullet})$  and  $A \sim B$  if  $A \cap B \neq \emptyset$ .

Refl. fails for  $A = \emptyset$



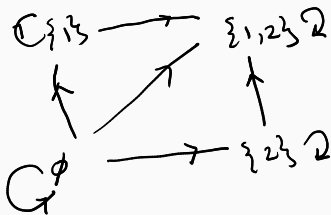
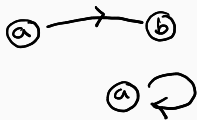
← why transitivity fails

# Graph of a relation

Let  $S$  be a set. It is helpful to visualise a relation by drawing a vertex for each element of  $S$  and an edge  $a \rightarrow b$  if  $(a, b)$  are related. The resulting diagram is called a *graph*.

Take  $S = \text{Pow}(\{1, 2\})$  and draw the graph of  $\subset$ .

$$S = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$$



# Graphical interpretation

How are the following properties reflected in the graph?

- Reflexivity

$\Leftrightarrow$  all self-loops



- Symmetry

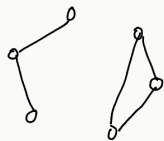


short hand

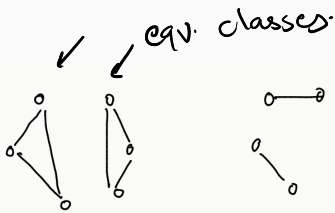


# Graph of an equivalence relation

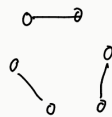
Which of the following is the graph of an equivalence relation?  
(Self-loops and arrows omitted)



X (1)



✓ (2)



✓ (3)

*eqv. rel*  
*"Clumped"*



✓



X



# Equivalence classes

Let  $\sim$  be the relation on  $\mathbb{R}^2$  defined by

$$(a, b) \sim (c, d) \text{ if } ab = cd.$$

Describe the equivalence classes.

