

WEEK 10 WORKSHOP  
MATH2301, SEMESTER 2, 2025

1. WARM-UP

1.1. **Problem.** Play some of the games we have learned: the subtraction game with different starting values and parameters, chomp, and nim.

1.2. **Problem.** Remember the rules of  $P$  vs  $N$  labelling.

- (1) If a state is labelled  $P$ , then which player has a winning strategy?
- (2) If a state is labelled  $N$ , then which player has a winning strategy?
- (3) If a state is a sink state (no outgoing arrows), we label it \_\_\_\_\_.
- (4) If a state has an outgoing arrow to a  $P$  state, we label it \_\_\_\_\_.
- (5) If all outgoing arrows of a state are  $N$ , we label it \_\_\_\_\_.

*Solution.*

- (1) The previous player (that is, the player who made a move to bring the game to the current state).
- (2) The next player (that is, the player who will make a move at the current state).
- (3)  $P$ .
- (4)  $N$ .
- (5)  $P$ .

1.3. **Problem.** Draw the game graph of some small games and label the game states as  $P$  or  $N$ . For example:

- (1) Eat 1 or 2 or 3 berries starting with 7 berries.
- (2) Eat any power of 2 (including  $2^0 = 1$ ) berries starting with 15 berries.
- (3)  $2 \times 2$  and  $2 \times 3$  chomp.

2. ADDING TWO GAMES

2.1. **Problem.** Explain what it means to add two games.

*Solution.* We play the two games side by side. A player is allowed to make a move in either of the two games. The player who has no valid moves loses.

2.2. **Problem.** Find two  $N$  games  $G_1$  and  $G_2$  such that  $G_1 + \text{Nim}(2, 3)$  and  $G_2 + \text{Nim}(2, 3)$  have different outcomes ( $N$  vs  $P$ ).

*Solution.* Take  $G_1 = \text{Nim}(2, 3)$ . Then  $G_1 + \text{Nim}(2, 3)$  is a  $P$ -game. Take  $G_2 = \text{Nim}(2)$ . Then  $G_2 + \text{Nim}(2, 3)$  is  $\text{Nim}(2) + \text{Nim}(3, 3)$  which has the same outcome as  $\text{Nim}(2)$  (why?), which is  $N$ .

3. WYT ROOKS

The game of Wyt rooks is played as follows. We start with two rooks placed on two squares of a chessboard. A move consists of moving one of the rooks any positive number of squares downward or leftward (but not both). Rooks can occupy the same square and move past each other. As usual, the player who cannot make a move loses.

3.1. **Problem.** Play this game with your group and determine some winning and losing positions.

*Solution :solution.* DIY.

3.2. **Problem.** This game is actually a nim game in disguise. Can you figure out how?

*Solution.* Number the chessboard so that  $(0, 0)$  is the bottom left square and  $(7, 7)$  is the top right square. With one rook at position  $(m, n)$ , the game is equivalent to  $(m, n)$  nim: moving left is taking from the first pile and moving down is taking from the second pile. Each additional rook gives you another 2-pile game of nim. So with several rooks it is just the sum of all these 2-pile nim games. For example, with rooks at  $(2, 3)$  and  $(3, 4)$ , the game is  $\text{Nim}(2, 3) + \text{Nim}(3, 4)$ , which is the same as  $\text{Nim}(2, 3, 3, 4)$ .

#### 4. EUCLID'S GAME

Euclid's game is defined as follows. The starting position is a pair of positive integers  $(a, b)$ . A move consists of subtracting a non-zero multiple of the smaller number from the larger number, ending up again with a pair of *positive* integers. In particular, any position of the form  $(n, n)$  is a P position.

**4.1. Problems.** Play this game with your group and determine some other non-trivial P and N positions.

Bonus/challenge: Can you find a pattern in who has a winning strategy?

*Solution.* Suppose that  $a \leq b$ . The answer to this problem is that  $(a, b)$  is an N position if and only if the ratio  $b/a$  is larger than the *golden ratio*, namely the number  $\phi = (1 + \sqrt{5})/2$ . To prove this, one has to show that from any position where the ratio is larger than the golden ratio, there is a move to a position whose ratio is smaller than the golden ratio, and from any position whose ratio is smaller than the golden ratio, every possible move is to a position where the ratio is larger than the golden ratio. The key is to observe that the number  $\phi$  satisfies the equation  $\phi^2 = \phi + 1$ . Happy to chat about details!