

# Games, graphs, and machines

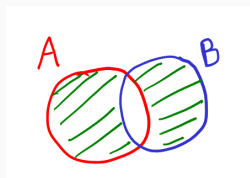
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# The symmetric difference

Suppose  $A$  and  $B$  are represented by the circles below. Using the operations of union, intersection, and difference, express the shaded set.

 $\cup$  $\cap$  $-$ 

$$= A \cup B - A \cap B$$

## Set operations

Is the following true or false:  $|A - B| = |A| - |B|$ . If it is true, explain why. If it is not true, give a counter-example.

$$A = \{1, 2\} \quad B = \{2, 3, 5\}$$

$$A - B = \{1\}$$

$$A = \{1, 2\} \quad B = \{2, P(\{1, 2, \dots, 100\})\}$$

$$A = \emptyset \quad B = \text{Any non-empty set}$$

$$A - B = \emptyset \quad \underline{\text{Fix}} \quad \bullet \quad |A - B| \geq |A| - |B|$$

• If  $B \subset A$  then

$$\bullet \quad |A - B| = |A| - |B \cap A|$$

# Function or not?

Do the following rules define functions?

Inj

Surj

✓ 1.  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(s) = s^2$ .

X

X

X 2.  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(s) = s/2$ .

✓ 3.  $f: \mathbb{Z} \rightarrow \mathbb{R}$  defined by  $f(s) = s/2$ .

X 4.  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(s) = \sqrt{s}$ .

5.  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(s) = \begin{cases} 1 & \text{if } s > 0 \\ -1 & \text{if } s < 0 \end{cases}$$

$$f(0) = 5$$

Non-neg sqrt

Fix  $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

$\sqrt{|s|}$

$\mathbb{R} \rightarrow \mathbb{C}$

↳ resolve

$\pm \sqrt{\quad}$  ambiguity

## Number of functions

Suppose  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3, 4\}$ .

1  
2  
3

4 possible  
— " —  
— " —

1. How many functions are there from  $A$  to  $B$ ?

12

64

24

81

## Number of functions

Suppose  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3, 4\}$ .

1. How many functions are there from  $A$  to  $B$ ?  $64$

2. How many of these are injective functions?

1		4 options	$\Rightarrow 24$
2		3 options	
3		2 options.	

## Number of functions

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1. How many functions are there from  $A$  to  $B$ ? 64

2. How many of these are injective functions? 24

3. How many of these are surjective functions? 0

## Number of functions (continued)

Suppose  $A$  has size  $n$  and  $B$  has size  $m$ .

How many functions are there from  $A$  to  $B$ ?

$$m^n$$

## Number of functions (continued)

Suppose  $A$  has size  $n$  and  $B$  has size  $m$ .

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## Number of functions (continued)

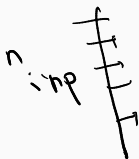
Suppose  $A$  has size  $n$  and  $B$  has size  $m$ .

How many functions are there from  $A$  to  $B$ ?  $m^n$

How many functions are injective? Assume  $n \leq m$

$$m (m-1) (m-2) \dots (m - (n-1))$$

How many functions are surjective? Assume  $n \geq m$



## Number of functions (continued)

Suppose  $A$  has size  $n$  and  $B$  has size  $m$ .

How many functions are there from  $A$  to  $B$ ?

How many functions are injective?

How many functions are surjective?

<https://oeis.org/A019538>

# The inverse function

Suppose  $f: S \rightarrow T$  is a bijection.

The inverse of  $f$  is the function  $g: T \rightarrow S$  defined by the property that if  $t = f(s)$  then  $s = g(t)$ .

1. Suppose  $f: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$  sends  $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$ . Find its inverse  $g$ .

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