# Games, graphs, and machines

In tropical algebra, the sum of two numbers is their minimum and the product of two numbers is their sum.

August 19, 2025

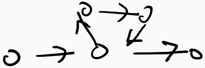
# Periodic?

Let $A$ be the boolean adjacency matrix of a graph.

Claim — The sequence of matrices $A, A^{*2}, A^{*3}, \ldots$ is periodic.

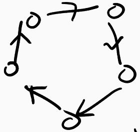
True or false?

2-periodic



3-periodic

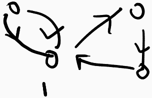
Ex



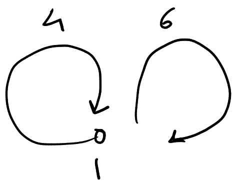
5-periodic

counter-ex

$$A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^2 = 0 \quad A^3 = 0 \quad \ldots$$



| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | … |
|---|---|---|---|---|---|---|---|---|
| $A^{*n}_{(1,1)}$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | … |




| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1,1) | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

"eventually periodic"

**Thm.** $A^0, A^{*1}, A^{*2}, A^{*3}, \ldots, A^{*7}, A^{*8}$ is eventually periodic.

**Proof**: $n \times n$ boolean matrices $\leftarrow$ finite ($2^{n^2}$)

$\Rightarrow$ There is a repeat ($\infty$ seq.)

$$A^{*n_1} = A^{*(n_1+N)}$$

Chaos $\leftarrow$

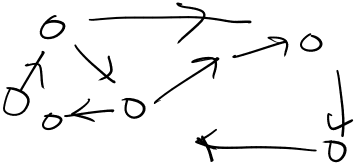

$\square$

Given a graph $\rightsquigarrow$ period of $A^n$

(number)

$\downarrow$

??



# Warm up with min/plus arithmetic

Remember that $\oplus = \min$ and $\odot = +$. Find

Numbers

$\shortparallel$

$\mathbb{R} \cup \{+\infty\}$

$$3: \quad \underbrace{2 \odot (\cancel{3 \oplus} 1)}_{3} \oplus \underbrace{3 \odot (\cancel{\infty \oplus} 2)}_{5}.$$

$\oplus$, $\odot$ obey the laws of $+$, $\cdot$

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c$$

<u>Boolean</u>

$+$ : OR

$\cdot$ : AND

<u>Tropical</u>

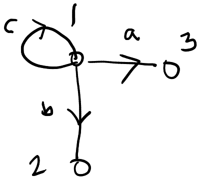
$\oplus$ <u>cost</u> of OR.

$\odot$ <u>cost</u> of AND

<u>Tropical</u>

$\oplus$ identity : $\infty$

$\odot$ identity : $0$



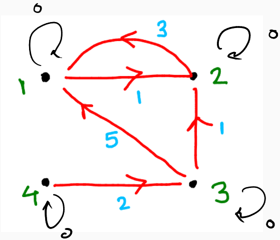

$$\begin{pmatrix} c & b & a \\ \infty & & \\ & & \end{pmatrix}$$

adj matrix,

"cost matrix"

# Weighted adjacency matrix

Write the min/plus weighted adjacency matrix of the graph. Assume that the loops have weight 0 (not shown).




$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$\begin{matrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & 0 & \infty \\ 7 & 3 & 2 & 0 \end{matrix}$$

## Min/plus powers

Find the min/plus square and ~~cube~~ of the adjacency matrix. What do its entries represent?

$$\begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & 0 & \infty \\ 7 & 3 & 2 & 0 \end{pmatrix}$$

$A^{\odot 2}$ = Shortest (least cost) paths of length 2.

$A^{\odot 3}$ = Least costly paths of length 3

$A^{\odot 3}, A^{\odot 4}, A^{\odot 5}, \dots$ stabilises.

# Why do min/plus powers give shortest paths?

For example, the third power:

$$\begin{aligned} A^3_{i,j} &= (A^2_{i,1} \odot A_{1,j}) \oplus (A^2_{i,2} \odot A_{2,j}) \oplus (A^3_{i,3} \odot A_{3,j}) \oplus (A^2_{i,4} \odot A_{4,j}) \\ &= \min(A^2_{i,1} + A_{1,j}, A^2_{i,2} + A_{2,j}, A^3_{i,3} + A_{3,j}, A^2_{i,4} + A_{4,j}) \end{aligned}$$

# When do we stop?

Assume:

1. we have all loops with weight 0,
2. all weights are non-negative.

**Theorem**
*Let $n$ be the number of vertices. Then* $A^{\odot(n-1)} = A^{\odot n} = A^{\odot(n+1)} = \cdots$.