ASSIGNMENT 6 (DUE 19 SEPTEMBER 2025)

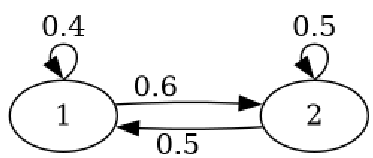
MATH2301, SEMESTER 2, 2025

1. General remarks

(1) The assignment is due on gradescope.
(2) Please read the academic integrity policy for assignments. Remember that if you want an extension, you must ask at least 24 hours ahead of the deadline.
(3) The words "show" and "prove" are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.
(4) If you are having trouble with any of the points, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

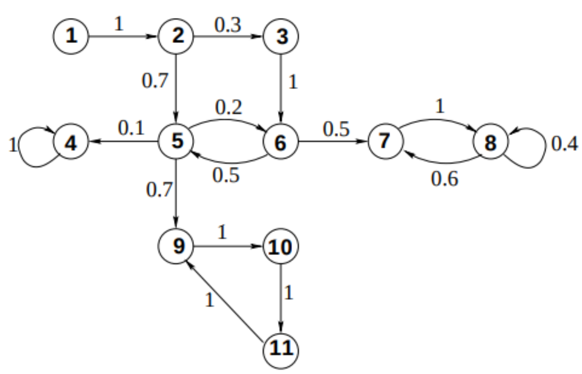
2. Problems

2.1. **Problem.** Consider the following Markov chain:



(1) What is the probability of transitioning from 1 to 1 after 2 steps?
(2) Let A be the transition matrix. Find $\lim_{k\to\infty} A^k$.

2.2. **Problem.** Consider the Markov chain



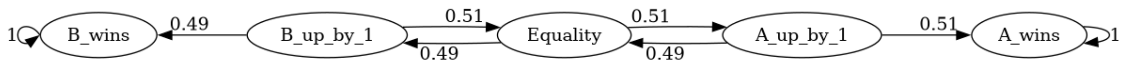
(1) Does the Perron-Frobenius theorem apply?
(2) Is the graph strongly connected?
(3) Define a relation $\sim$ on the set of vertices by saying $v \sim w$ if
  (a) $v = w$ or
  (b) there is a path from v to w and a path from w to v.
  Convince yourself (but do not turn in) that this is an equivalence relation. Write down the equivalence classes.

2.3. **Problem.** Let $S = \mathbf{Z}/5\mathbf{Z} - \{\overline{0}\}$. On the board, I have written the element $\overline{1} \in S$. At every step, I consider the number a on the board. If a is a square modulo 5, I erase a and replace it with one of the two square roots (chosen at random with equal probability). If a is not a square modulo 5, I erase a and write $\overline{2} \cdot a$.

(1) Draw a weighted directed graph that models the game.
(2) Let A be the transition matrix. Use the Perron-Frobenius theorem to find $\lim_{k\to\infty} A^k$.

2.4. **Problem.** In many games, it is common to have a "win by 2" rule. Initially, both players have score 0, say. They play for a point until one of them leads by 2, when the leader is declared the winner.

Suppose the players are A and B. Let us say that A has a 51% chance of winning a point and B has a 49% chance of winning a point (so A is slightly more skilled). A "win by 2" game is modelled by the following Markov chain:



(1) Starting at `Equality`, what is the probability that A wins? What is the probability that B wins? (Feel free to use a computer to compute large matrix powers if you need to.)
(2) Suppose we change the game to a "win by 3" game. Draw the corresponding Markov chain. Now what is the probability that A wins?
(3) (Not to be turned in) Experiment with "win by 4", "win by 5", and so on. Observe that increasing the winning margin magnifies the original skill-advantage.