Games, graphs, and machines

https://miachenmtl.github.io/random-jumps/

August 26, 2025

Three possibilities

Let A be the transition matrix of a Markov chain.

1. $\lim_{n\to\infty} A^n$ exists and has the same rows.
2. $\lim_{n\to\infty} A^n$ exists but has different rows.
3. $\lim_{n\to\infty} A^n$ does not exist.

Understanding large powers

It turns out that the sunny/rainy matrix A can be written as

$$A = EDE^{-1},$$

where $E = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}.$

Use this to understand A^k for large k.

When do large powers converge?

Suppose $A = EDE^{-1}$, where D is diagonal. When will A^k converge (to a matrix with finite entries) as k grows?

The Perron-Frobenius theorem

Theorem
Let A be the transition matrix of a Markov chain. Suppose there exists an n such that for every i and j, there is a path of length n from state i to state j. Then

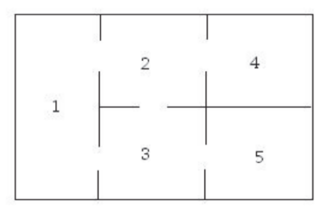
1. $\lim_{n\to\infty} A^n$ *exists.*
2. *The limiting matrix has identical rows.*
3. *The limiting row vector v is the unique vector whose entries sum to 1 and which satisfies the equation*

$$vA = v.$$

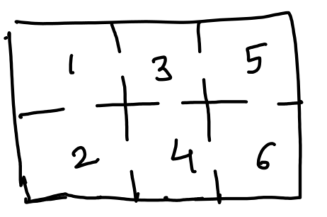
Does it apply?

To the random walk of the rats?

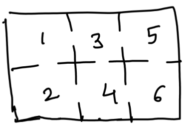
- with cheese in rooms 4/5?
- without?



A slightly different maze



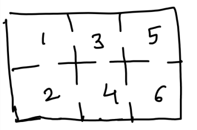
A slightly different maze




```
A = matrix([[0, 1/2, 1/2, 0, 0,0],
            [1/2,0,0,1/2,0,0],
            [1/3,0,0,1/3,1/3,0],
            [0,1/3,1/3,0,0,1/3],
            [0,0,1/2,0,0,1/2],
            [0,0,0,1/2,1/2,0]])
(A^(100)).n(10)
```

```
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
```

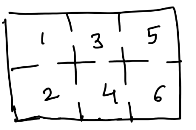
A slightly different maze




```
(A^(101)).n(10)

[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

A slightly different maze




```
(A^(101)).n(10)
```

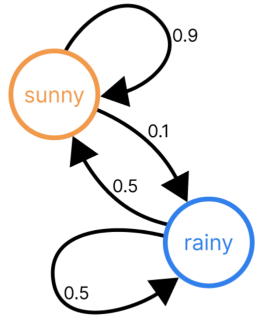
```
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

```
A.eigenvalues()
```

```
[1, 1/2, 1/6, -1/6, -1/2, -1]
```

Does it apply?

To the weather forecaster?



Why does does Perron-Frobenius hold?

The crucial point is that under the hypotheses:

- only one eigenvalue is 1
- all the others have absolute value < 1