WEEK 11 WORKSHOP

MATH2301, SEMESTER 2, 2025

1. Nim sum

1.1. **Problem.** Find the nim sum of the following nim games and using it, find all winning (first) moves.

(1) $\mathrm{Nim}(2,3,4)$
(2) $\mathrm{Nim}(2,9,10)$

1.2. **Solution.**

(1) The nim sum in binary is 101. Xoring this with 100 results in a smaller number, 1. So the only winning move is $(2,3,4) \mapsto (2,3,1)$.
(2) The nim sum in binary is 1. Xoring this with 9 results in a small number, 8. So the only winning move is $(2,9,10) \mapsto (2,8,10)$.

1.3. **Problem.** Construct 5-pile nim games that has one/three/five possible winning first moves. Is there a nim game that has two possible winning first moves? Why?

Solution. There are many examples, and I will leave it to you to construct them. Here is one with three possible winning first moves: 3,3,4,5,6.

We know how to find the winning moves. Given the nim sum, look at its most significant digit (leftmost digit). The piles that have a 1 at this place produce a smaller number when xor'ed with the nim. The piles that have a 0 at this place produce a bigger number. There must be an odd number of piles with 1 at this place (because the nim-sum has a 1 there). So it is not possible to have an even number of winning moves.

1.4. **Problem.** Consider $\mathrm{Nim}(4,5,6)$, which has nim sum $100 \oplus 101 \oplus 110 = 100$, which is 4. Find moves that take it to a state with nim sum $0,1,2,3$. Is there a move that takes it to a state with nim sum 4?

Repeat the procedure with another nim, say $\mathrm{Nim}(3,3,5$. It has nim sum 5. Find moves that take it nim sum $0,1,2,3,4$. Is there a move that takes it to a state with nim sum 5?

Solution. DIY.

1.5. **Problem.** Convince your friends that the following is true: if a nim game has nim sum n, then it is possible to take it to a game with nim sum $0,1,2,\ldots,n-1$, but not n.

(If you are stuck, do the other problems and then come back to this one.)

Solution. Let m be one of $0,1,2,\ldots,n-1$. Consider $b = n \oplus m$ (in binary). Then $m = n \oplus b$. Since $m < n$, we know that n must have 1 at the most significant (leftmost) digit of b. Since n has 1 at this digit, there is at least one pile p that has 1 at this digit. Let $p' = p \oplus b$. Since p has 1 at the most significant digit of b, we conclude that $p' < p$. Taking p to p' is then a legal move that takes the nim sum from n to m.

It is not possible to keep the same nim sum for the same reason that it is not possible to move from a game with nim sum 0 to a game with nim sum 0. Changing a single pile necessarily changes the nim sum.

2. Divisor chomp

The game of divisor chomp is played as follows. We start with all positive divisors of a number N written on the board (except 1). On their turn, a player erases a number d together with all d' such that d divides d'. For example, for $N = 12$, we will start with $2,3,4,6,12$ on the board. If the first player chooses $d = 6$, then 6 and 12 are erased. Then, if the second player chooses $d = 3$, then only 3 is erased. And so on. The player who erases the last number wins.

2.1. **Problem.** Play divisor chomp with a few values of N.

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2.2. **Problem.** Divisor chomp with $N = 200$ is actually 2×3 chomp in disguise. Why? Can you similarly interpret divisor chomp with $N = 900$?

Solution. We have $200 = 2^3 \times 5^2$. So the divisors of 200 are represented by $2^a 5^b$ with $0 \leqslant a \leqslant 3$ and $0 \leqslant b \leqslant 2$. This is a 2×3 grid; we erase $(0, 0)$ because the divisor 1 is excluded. Now $2^a 3^b$ divides $2^{a'} 3^{b'}$ if and only if $a \leqslant a'$ and $b \leqslant b'$. So, in the correct orientation, erasing (a, b) also erases all grid points in the below/right quadrant. Those are exactly the rules of chomp.

For $N = 900$, we have $900 = 2^2 \times 3^2 \times 5^2$. So poset-chomp here is 3 dimensional chomp played on a $2 \times 2 \times 2$ grid (excluding one corner, which we say is "poisoned"). In 3 dimensions, the below/right quadrant is replaced by an octant.

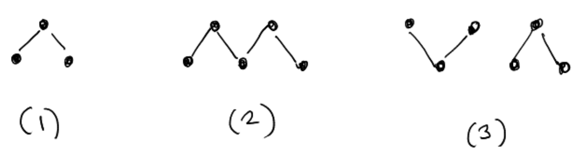
3. Poset chomp

The game of poset chomp is played as follows. We start with a poset drawn on the board. On their turn, a player erases an element d together with all d' such that $d \preceq d'$. The player who erases the last element of the poset wins.

3.1. **Problem.** The usual chomp is an example of poset chomp for a particular poset. Which poset?

Solution. It is the product poset $\{0, \dots, n\} \times \{0, \dots, m\}$ minus the element $(0, 0)$.

3.2. **Problem.** These are the Hasse diagrams of some posets. Suppose we play poset chomp starting with those posets. Label the starting state as N or P.



Solution. It is possible to get away without having to analyse the entire game tree. In (1), eating the top vertex leaves two copies of the same poset. This means the game is the sum of two identical games, which is P. So (1) is N.

In (2), the same logic applies to the move that takes the middle vertex on the bottom row. So (2) is also N.

For (3) also I got N. I will let you work it out.