WEEK 6 WORKSHOP

MATH2301, SEMESTER 2, 2025

1. THE PERRON–FROBENIUS PROPERTY

Let G denote a Markov chain with transition matrix A.

1.1. **Problem.** When does the Perron–Frobenius theorem (PFT) apply? Give your answer in terms of

(1) the existence/non-existence of paths,
(2) powers of A,
(3) powers of the boolean adjacency matrix B.

1.2. **Problem.** Construct examples of graphs G (omit the probabilities) where

(1) PFT applies,
(2) PFT does not apply,
(3) the graph is strongly connected and yet PFT does not apply.

2. THE GCD CONDITION

Recall that a convenient way to verify that PFT applies is:

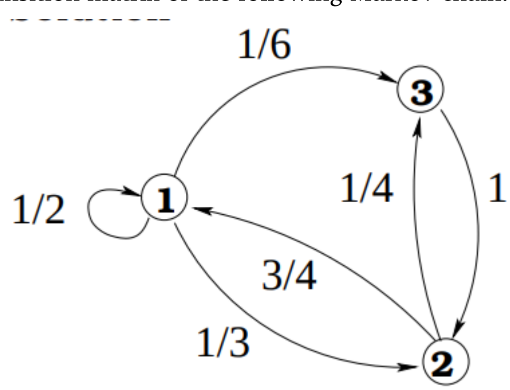
- make sure the graph is strongly connected,
- find a vertex v and directed cycles based at v of length a and b with $\gcd(a, b) = 1$.

2.1. **Problem.** Let G be the Markov chain of snakes and ladders on a 10×10 board where we continue from 0 if we go beyond 100. (So rolling a 4 at 98 takes us to 2). Verify the two conditions above.

2.2. **Problem.** Does your argument work if we change the size of the board? What if we change the 6-sided die to an 8-sided die?

3. COMPUTING THE STEADY STATE

3.1. **Problem.** Let A be the transition matrix of the following Markov chain. Use PFT to find $\lim_{n\to\infty} A^n$.



Date: August 25, 2025.

4. Menedlian genetics

Gregor Mendel observed a particular gene in garden peas that exists in two types: G or g. Each pea plant has a pair of genes. So a plant could be of type GG, or $Gg = gG$, or gg. We take a plant and fertilise it with a plant of type Gg to produce an offspring. We do the same process starting with the offspring (always mating with a Gg plant), and continue. The offspring inherits one gene from each parent with equal probability. So, for example, if we fertilise GG with Gg, then the offspring will be GG with probability $1/2$ and Gg with probability $1/2$. If we fertilise Gg with Gg, the offspring will be GG with probability $1/4$, $Gg = gG$ with probability $1/2$, and gg with probability $1/4$.

4.1. **Problem.** Taking $\{GG, Gg, gg\}$ as the vertex set, describe the Markov chain.

4.2. **Problem.** Decide if PFT applies.

4.3. **Problem.** Write the transition matrix A and compute the first few powers. Observe the row corresponding to Gg. What do you notice? Can you interpret your result? What about the other two rows?

5. Random walks (if time permits)

Let G be a directed graph. Let A be the transition matrix of the random walk on G.

5.1. **Problem.** Suppose G is symmetric, connected, and has a loop at every vertex. Show that PFT applies. Can you find the limiting distribution?

5.2. **Problem.** Try relaxing the conditions above and explore what happens (use a computer).

6. The GCD condition (if time permits)

The reason the GCD condion works is the following theorem.
Theorem: Let a, b be positive integers with $\gcd(a, b) = 1$. Then any $n > ab$ can be written as a sum of a's and b's.

(1) Try to prove it for $a = 3$ and $b = 4$.
(2) Try with $a = 3$ and $b = 5$.
(3) Try with any a and $b = a + 1$.
(4) For the general case, let r be the remainder when n is divided by a.
   - If $r = 0$, we can simply write $n = a + a + \cdots + a$ (no b's needed).
   - If $r \equiv b \pmod{a}$, what would you do?
   - If $r \equiv 2b \pmod{a}$, what would you do?
   - If $r \equiv 3b \pmod{a}$, what would you do?
   - Can you generalise?