Games, graphs, and machines

Relations

July 23, 2025

A relation between A and B is a subset of $A \times B$.

	b_1	b_2	b_3	$\cdots$
a_1	✓	✗	✓	
a_2	✓	✓	✗	
a_3				$\ddots$
$\vdots$				
a_{100}				

elts of B

elts of A

Locations of ✓ $\cap$ $A \times B$

Relations

A relation between A and B is a subset of $A \times B$.

$R \subset A \times B$

$A =$ ANU Students.

$B = \{2025, 2024, \ldots, 2000\}$

(Student, year) $\in R$

if student was in ANU in year.

The number of relations

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. How many relations are there between A and B?

$\parallel$

Subset of

$A \times B$

12 125 4096

$\Rightarrow$ 2^{12} subsets of a 12 elts set

$2^6 \cdot 2^6$

$2^3 \cdot 2^3$

$$\begin{array}{r} 64 \\ 64 \\ \hline 4096 \end{array}$$

25

Reflexive relations

We say that a relation $R \subset S \times S$ is *reflexive* if for all $s \in S$, we have $(s, s) \in S$.

	s_1	s_2	…
s_1	✓		
s_2		✓	
⋮			✓ … ✓

Are the following relations reflexive?

✓ 1. $\leq$ on $\mathbb{R}$ $\quad (\{(x,y) \mid x \leq y\} \subset \mathbb{R}^2)$

✗ 2. $<$ on $\mathbb{R}$

✓ 3. the relation R on $\mathbb{Z}$ defined by $(a, b) \in \mathbb{Z}^2$ if 2 divides $a + b$.

✗ 4. Same as above, but with 2 replaced by 3.

$A = \{1,2,3\}$ How many refl. rel on A ?

2^6

	1	2	3
1	✓		
2		✓	
3			✓

6 remaining can be ✓, ✗

Symmetric relations

We say that a relation $R \subset S \times S$ is *symmetric* if for all $s \in S$ and $t \in S$, if $(s, t) \in R$ then $(t, s) \in R$.

Find a relation on $\mathbb{Z}$ that is symmetric and one that is not symmetric.

Ex. Symmetric

1. $(a,b) \in R$ if 7 divides $a-b$
2. -||- if 7 divides $a+b$
3. $\emptyset$

Non-symmetric

1. $(a,b) \in R$ if a divides b
2. $(a,b) \in R$ if a is a power of 2.

Symmetric relations

We say that a relation $R \subset S \times S$ is *symmetric* if for all $s \in S$ and $t \in S$, if $(s, t) \in R$ then $(t, s) \in R$.

Fi~~nd a relation on $\mathbb{Z}$ that is symmetric and one that is not symmetric~~.

$$A = \{1, 2, 3\}$$

How many symmetric relations?

Also 2^6

	1	2	3
1	?	□	○
2	■	?	△
3	●	▲	?

Transitive relations

We say that a relation $R \subset S \times S$ is *transitive* if for all $a, b, c \in S$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Find a relation on $\mathbb{R}$ that is transitive and one that is not transitive.

<u>Example</u>

0. $\emptyset$
1. $(a,b) \in R$ if a is a power of 2.
2. $\leq$ inequality
3. $<$ strict inequality
4. $=$ equality

<u>Non-example</u>

1. $(a,b) \in R$ if ab is an even integer.

$a=5 \quad b=2 \quad c=3$

2. $\neq$ unequality

Transitive relations

We say that a relation $R \subset S \times S$ is *transitive* if for all $a, b, c \in S$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

~~Find a relation on $\mathbb{R}$ that is transitive and one that is not transitive.~~

$A = \{1,2,3\}$

How many transitive relations on an n elt set.

???

Probably an OEIS

online encyclopedia of integer seq.

Transitive relations (continued)

Are the following relations transitive?

✓ 1. $\leq$ on $\mathbb{R}$
✓ 2. $<$ on $\mathbb{R}$
✓ 3. the relation R on $\mathbb{Z}$ defined by $(a, b) \in \mathbb{Z}$ if 2 divides $a + b$.
✗ 4. Same as above, but with 2 replaced by 3.

$(2,1) \in R \qquad (1,2) \in R$

$(2,2) \notin R.$

Input/Output relation

Consider $R \subset \mathbb{R} \times \mathbb{R}$ defined by

$$R = \{(x, y) \mid x^3 - xy + x - 1 = 0\}.$$

Is R the input/output relation of a function $f : \mathbb{R} \to \mathbb{R}$?