WEEK 5 WORKSHOP

MATH2301, SEMESTER 2, 2025

1. Boolean arithmetic with negative numbers

Let us see what happens with boolean arithmetic if we allow negative numbers.

Consider the equivalence relation on $\mathbf{R}$ defined by $a \sim b$ if both a and b are zero, or if both a and b have the same sign.

1.1. **Problem.** What are the equivalence classes?

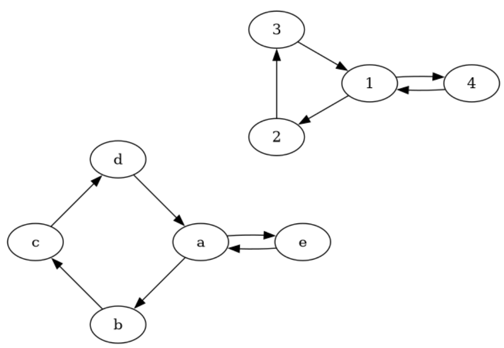
1.2. **Problem.** Try to define $+$ and $\cdot$ on the equivalence classes by following the same procedure as before. To compute $A+B$ for two equivalence classes A and B, we pick a number in A, a number in B, add them, and then take the equivalence class of the sum. Similarly for $\cdot$. Does this procedure define a consistent $+$ or $\cdot$ on the equivalence classes?

2. Oscillation versus stabilisation

2.1. **Problem.** Construct a boolean matrix whose boolean powers oscillate with a cycle of 3.

2.2. **Problem.** We saw in the lecture that the powers of any boolean matrix A are eventually periodic.

What is the period in the following two examples?



2.3. **Problem.** Let A be any boolean matrix. True or false: the accumulated powers $I + A + A^2 + A^3 + \cdots + A^n$ stabilise. That is, after a certain n, the value of $I + A + \cdots + A^n$ does not depend on n.

3. Powers versus accumulated powers

3.1. **Problem.** Convince yourselves, that in boolean algebra, we have

$$(I + A)^n = I + A + A^2 + \cdots + A^n.$$

Start by taking $n = 1, 2, 3, \ldots$. It may be helpful to notice that $A + A = A$.

3.2. **Problem.** Let A be the adjacency matrix of a graph. Let B be the adjacency matrix of the same graph with all self-loops added. Explain the phenomenon: the n-th power of B is the n-th accumulated power of A.

That is,

$$B^n = I + A + \cdots + A^n.$$

Date: August 18, 2025.