

## ASSIGNMENT 2 (DUE 8 AUGUST 2025)

### MATH2301, SEMESTER 2, 2025

#### 1. GENERAL REMARKS

- (1) The assignment is due on gradescope.
- (2) Please read the academic integrity policy for assignments.
- (3) The words “show” and “prove” are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.
- (4) If you are having trouble with any of the points, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

#### 2. PROBLEMS

2.1. **Problem.** Let  $S$  be the set of squares on a standard  $8 \times 8$  chessboard. Consider the following relations on  $S$ .

(1)  $B = \{(s, t) \in S \times S \mid t \text{ is reachable from } s \text{ by a sequence of zero or more bishop moves.}\}$

(2)  $R = \{(s, t) \in S \times S \mid t \text{ is reachable from } s \text{ by a sequence of zero or more rook moves.}\}$

(3)  $K = \{(s, t) \in S \times S \mid t \text{ is reachable from } s \text{ by a sequence of zero or more knight moves.}\}$

It turns out that all of these are equivalence relations (you can check this privately, but you don't have to justify it). In each case, determine how many equivalence classes there are.

2.2. **Problem.** Consider modular arithmetic with the modulus  $d = 10$ . For each equivalence class  $[x] \in \mathbf{Z}/d\mathbf{Z}$ , determine whether or not  $[x]$  has a multiplicative inverse, and if yes, find the inverse. That is, figure out whether there is some  $[y]$  such that  $[x] \cdot [y] = [1]$ .

*Bonus (not to be turned in):* Can you find a pattern here? When does  $[x]$  have a multiplicative inverse? Does a number ever have more than one inverse?

2.3. **Problem.** Show that if  $[3x] = [5]$  modulo 7, then  $[x] = [4]$ .

2.4. **Problem.** This is an exploratory question. The submission will ask for a few results of your exploration, but you should explore to your heart's content. Feel free to experiment by hand, using a calculator, or a computer. Using a program that you did not write yourself is also OK (usually it is not), but if it is based on a randomised algorithm (like a large language model), make sure that the answers are correct.

The broad question is: how many square roots does 1 have in  $\mathbf{Z}/n\mathbf{Z}$  for various  $n$ ?

- Are there  $n$  for which there is only one square root?
- Are there  $n$  for which there are exactly two square roots?
- Are there  $n$  for which there are more than two? More than 3? More than 4? More than 10? More than 100?
- Is the number of square roots always even or always odd?
- Can you observe any patterns?
- Can you explain the patterns you observe?