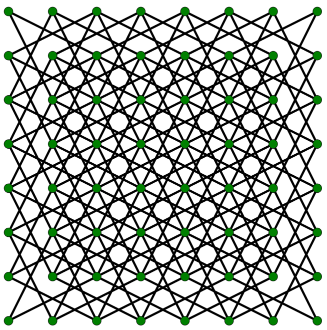
Games, graphs, and machines

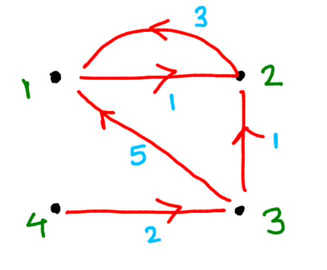


August 20, 2025

Min/plus powers = least costly paths

Theorem

Let A be the min/plus adjacency matrix of a graph. Then $A^{\odot n}_{i,j}$ is the cost of the least costly path of length n from i to j.



$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

(and loops of cost 0)

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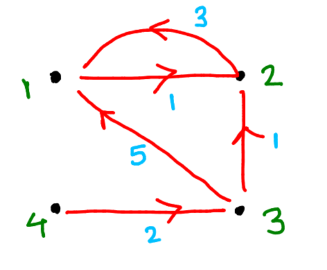

(and loops of cost 0)

$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$A^{\odot 2} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & \infty & \infty \\ 7 & 3 & 2 & 0 \end{pmatrix}$$

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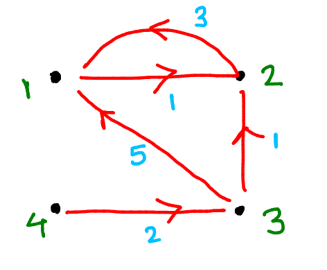

(and loops of cost 0)

$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$A^{\odot 3} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & \infty & \infty \\ 6 & 3 & 2 & 0 \end{pmatrix}$$

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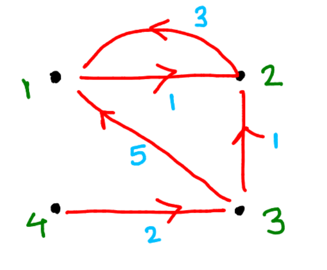

(and loops of cost 0)

$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$A^{\odot 4} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & \infty & \infty \\ 6 & 3 & 2 & 0 \end{pmatrix}$$

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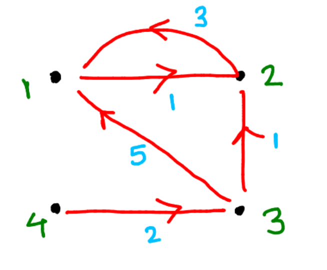

(and loops of cost 0)

$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$A^{\odot 5} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & \infty & \infty \\ 6 & 3 & 2 & 0 \end{pmatrix}$$

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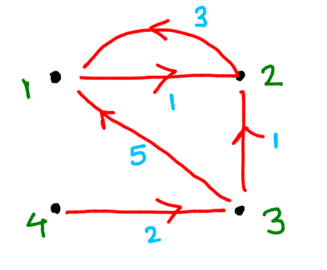
Let A be the min/plus adjacency matrix of a graph. Then $A^{\odot n}_{i,j}$ is the cost of the least costly path of length n from i to j.



(and loops of cost 0)

$$A = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix}$$

$$A^{\odot n} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & \infty & \infty \\ 6 & 3 & 2 & 0 \end{pmatrix}$$

Why do min/plus powers give shortest paths?

For example, the third power:

$$
\begin{aligned}
A^3_{i,j} &= (A^2_{i,1} \odot A_{1,j}) \oplus (A^2_{i,2} \odot A_{2,j}) \oplus (A^3_{i,3} \odot A_{3,j}) \oplus (A^2_{i,4} \odot A_{4,j}) \\
&= \min(A^2_{i,1} + A_{1,j}, A^2_{i,2} + A_{2,j}, A^3_{i,3} + A_{3,j}, A^2_{i,4} + A_{4,j})
\end{aligned}
$$

Why do min/plus powers stabilise?

1. Assume we have all loops with cost 0.
   Then $A^{\odot n}$ also calculates the lowest cost for paths of length up to n.

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2. Assume all costs are non-negative.
   If we have N vertices, then the lowest cost is achieved by a path of length up to N.

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1. Assume we have all loops with cost 0.
   Then $A^{\odot n}$ also calculates the lowest cost for paths of length up to n.
2. Assume all costs are non-negative.
   If we have N vertices, then the lowest cost is achieved by a path of length up to N.

Theorem
Let N be the number of vertices. If all loops are present with cost 0 and all costs are non-negative, then $A^{\odot N} = A^{\odot(N+1)} = \cdots$.