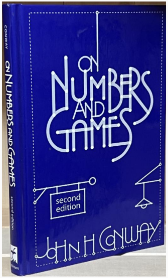
Games, graphs, and machines



October 21, 2025

Why is Grundy$(G + H) =$ Grundy$(G) \oplus$ Grundy(H)?

- Suppose Grundy$(G) = 10110$ and Grundy$(H) = 11100$.
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We need two facts:

1. No child of $G + H$ has label x.
2. For every $y < x$, a child of $G + H$ has label y.

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A child of $G + H$ is

- $G' + H$ for $G \rightsquigarrow G'$ or
- $G + H'$ for $H \rightsquigarrow H'$.

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- To take the Grundy value of $G + H$ to y, can
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- At least one of these leads to a smaller value, and hence is possible.

Partial games

Dominos – Left player places horizontally. Right player places vertically.

Four outcomes

- Left has a winning strategy
- Right has a winning strategy
- First player has a winning strategy
- Second player has a winning strategy

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- and more!