ASSIGNMENT 4 (DUE 22 AUGUST 2025)

MATH2301, SEMESTER 2, 2025

1. General remarks

(1) The assignment is due on gradescope.
(2) Please read the academic integrity policy for assignments.
(3) The words "show" and "prove" are synonyms. You may not be used to writing formal mathematical proofs, which is OK. Write a justification in plain language that would convince the reader.
(4) If you are having trouble with any of the points, come and discuss with me in office hours. It is part of my job to help you understand this stuff, so please use my time!

2. Problems

2.1. **Problem.** Consider a graph whose adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the number of paths of length 4 from 1 to 3.

2.1.1. *Solution.* We solve this by computing the A^4 and taking the entry in the spot $(1, 3)$.

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^4 = (A^2)^2 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

So the answer is 10.

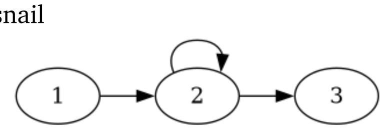
2.2. **Problem.** Find (without explicit calculation) an example of a 4×4 nonzero adjacency matrix such that all powers of this matrix beyond the 10th power are zero.

2.2.1. *Solution.* We do this by drawing a graph where there are no paths of length 10 or higher between any pair of vertices. For example, we can consider

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that if the 8th power is not zero, then some entry (i, j) is nonzero, so there is a length eight path from i to j. But there are only four vertices, so there must be some loop in this path! By repeating the loop several times, we can get longer and longer paths from i to j, so there must certainly be paths of length beyond 10. But all powers beyond the 10th power are zero, so this can only happen if the 8th power of the matrix was zero to begin with.

2.3. **Problem.** Let G be the directed snail



Let A be the adjacency matrix of G. Describe all positive powers of A.

2.3.1. *Solution.* We have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $n \geqslant 2$ be a positive integer. Then there is a unique path of length n from 1 to 2, of the form

$$1 \to 2 \to \cdots \; n-1 \text{ loops } \cdots \to 2.$$

There is a unique path of length n from 1 to 3, of the form

$$1 \to 2 \to \cdots \; n-2 \text{ loops } \cdots \to 2 \to 3.$$

There is a unique path of length n from 2 to 2, of the form

$$2 \to \cdots \; n \text{ loops } \cdots \to 2.$$

There is a unique path of length n from 2 to 3, of the form

$$2 \to \cdots \; n-1 \text{ loops } \cdots \to 2 \to 3.$$

There are no other paths of length n. So, for $n \geqslant 2$, we have

$$A^n = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

2.4. **Problem.** This is an exploratory problem.

We can classify graphs according to how quickly the entries in the powers of the adjacency matrix grow. Denoting by A the adjacency matrix, we have

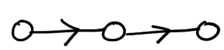
(1) graphs such that $A^n = 0$ for large enough n.
(2) graphs such that the entries of A^n are bounded by a constant,
(3) graphs such that the entries of A^n are bounded by a polynomial function of n,
(4) graphs such that the entries of A^n grow exponentially as n grows.

Find examples of graphs in each class.

Given a graph, is there a way to easily tell where falls among the classes above?

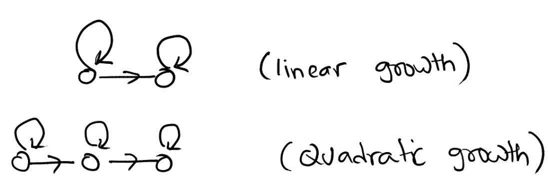
2.4.1. *Solution.* Here are some examples.

(1) Powers eventually zero.



(2) Powers bounded by a constant.

(3) Powers of polynomial growth.



(4) Powers of exponential growth.

I do not know a satisfactory graph theoretic characterisation.